STABLE EQUIVALENCE OF MORITA TYPE AND FROBENIUS EXTENSIONS

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Abstract. A.S. Dugas and R. Martínez-Villa proved in [4, Corollary 5.1] that if there exists a stable equivalence of Morita type between the k-algebras Λ and Γ, then it is possible to replace Λ by a Morita equivalent k-algebra Δ such that Γ is a subring of Δ and the induction and restriction functors induce inverse stable equivalences. In this note we give an affirmative answer to a question of Alex Dugas about the existence of a Γ-coring structure on Δ. We do this by showing that Δ is a Frobenius extension of Γ.

As in [4], we will assume throughout that the algebras Λ and Γ are finite dimensional over a field k and have no semisimple blocks.

The algebras Λ and Γ are said to be stably equivalent if the categories of finitely generated modules modulo projectives for Λ and Γ are equivalent (see [1]).

A pair of left-right projective bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ is said to induce a stable equivalence of Morita type between Λ and Γ if we have the following isomorphisms of bimodules:

$$ {}_\Lambda M \otimes_\Gamma N_\Lambda \simeq {}_\Lambda \Lambda_\Lambda \oplus {}_\Lambda P_\Lambda \quad \textit{and} \quad {}_\Gamma N \otimes_\Lambda M_\Gamma \simeq {}_\Gamma \Gamma_\Gamma \oplus {}_\Gamma Q_\Gamma $$

where ${}_\Lambda P_\Lambda$ and ${}_\Gamma Q_\Gamma$ are projective bimodules (see [2]).

We begin by stating the result of Dugas and Martínez-Villa mentioned in the abstract:

Theorem 1. *(see* [4, Corollary 5.1]*) Let Λ and Γ be finite dimensional k-algebras whose semisimple quotients are separable. If at least one of them is indecomposable, then the following are equivalent:*
(1) There exists a stable equivalence of Morita type between Λ and Γ.
(2) There exists a k-algebra Δ, Morita equivalent to Λ, and an injective ring homomorphism $\Gamma \hookrightarrow \Delta$ such that the restriction and induction functors are exact and induce inverse stable equivalences.
(3) There exists a k-algebra Δ, Morita equivalent to Λ, and an injective ring homomorphism $\Gamma \hookrightarrow \Delta$ such that

$$ {}_\Gamma \Delta_\Gamma = {}_\Gamma \Gamma_\Gamma \oplus {}_\Gamma P_\Gamma \quad \textit{and} \quad {}_\Delta \Delta \otimes_\Gamma \Delta_\Delta \simeq {}_\Delta \Delta_\Delta \oplus {}_\Delta Q_\Delta $$

for projective bimodules ${}_\Gamma P_\Gamma$ and ${}_\Delta Q_\Delta$.

We recall now the definition of Frobenius extension, and its dual notion, Frobenius coring.

Definition 2. *(see* [5]*) Let $i : R \longrightarrow S$ be a ring homomorphism. Then S/R is called a Frobenius extension if one of the following equivalent conditions is satisfied:*

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(1) *S is finitely generated and projective as a right R-module and $Hom_R(S,R)$ and S are isomorphic as (R,S)-bimodules;*
(2) *there exists a Frobenius system (e,ε), consisting of*

$$e = e^1 \otimes e^2 \in (S \otimes_R S)^S = \{e^1 \otimes e^2 \in S \otimes_R S \mid se^1 \otimes e^2 = e^1 \otimes e^2 s, \forall s \in S\}$$

and $\varepsilon:\ S \to R$ an R-bimodule map such that $\varepsilon(e^1)e^2 = e^1\varepsilon(e^2) = 1$.

For the proof of the equivalence of the two conditions, see for example [3, Theorem 28].

Definition 3. *(see [7]) If R is a ring, a coring is a comonoid in the monoidal category of R-bimodules. So a coring consists of an R-bimodule C, together with a coassociative comultiplication $C \longrightarrow C \otimes_R C$ and counit $C \longrightarrow R$ which are both R-bimodule maps. C is called a Frobenius R-coring if there exists a Frobenius system $(\theta, 1)$, consisting of an element $1 \in C$ and an R-bimodule map $\theta:\ C \otimes_R C \to R$ satisfying the conditions*

$$c_{(1)}\theta(c_{(2)} \otimes d) = \theta(c \otimes d_{(1)})d_{(2)} \text{ and } \theta(c \otimes 1) = \theta(1 \otimes c) = \varepsilon(c).$$

Let $(S, m, 1, e, \varepsilon)$ be a Frobenius extension of R, and consider $\Delta:\ S \to S \otimes_R S$, $\Delta(s) = se = es$. An easy verification shows that $(S, \Delta, \varepsilon, \theta = \varepsilon \circ m, 1)$ is a Frobenius coring.
Conversely, if $(C, \Delta, \varepsilon, \theta, 1)$ is a Frobenius R-coring, then $(C, m, 1, \Delta(1), \varepsilon)$ is a Frobenius extension. Here $m:\ C \otimes_R C \to C$, $m(c \otimes d) = c_{(1)}\theta(c_{(2)} \otimes d) = \theta(c \otimes d_{(1)})d_{(2)}$.
These two assertions basically tell us that Frobenius extension structures on an R-bimodule M correspond bijectively to Frobenius R-coring structures on M.
Let S be a Frobenius extension. Then the categories $\mathcal{M}_S$ and $\mathcal{M}^S$ are isomorphic: on a right S-module, we define a right S-coaction by $\rho(m) = me^1 \otimes e^2$. On a right S-comodule, we define a right S-action $ms = m_{[0]}\varepsilon(m_{[1]}s)$.
The restriction functor $G : \mathcal{M}_S \to \mathcal{M}_R$ has a left adjoint, the induction functor F; the forgetful functor $\mathcal{M}_S \to \mathcal{M}_R$ has a right adjoint. These functors are compatible with the above isomorphism. This implies that G is at the same time a left and a right adjoint of F.

Definition 4. *(see [6] or [3, p.91]) Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor. If there exists a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ which is at the same time a right and a left adjoint of F, then we call F a* Frobenius functor*, and we say that (F,G) is a* Frobenius pair *for $\mathcal{C}$ and $\mathcal{D}$.*

Remark 5. *(see [5] or [3, Theorem 28, p.103]) Let $i : R \longrightarrow S$ be a ring homomorphism, F the induction functor and G the restriction functor. If S/R is a Frobenius extension, then we have seen above that (F,G) is a Frobenius pair; in fact, it can be shown that the converse also holds: (F,G) is a Frobenius pair if and only if S/R is a Frobenius extension.*

We can now state and prove our result. Assertion (3) gives an affirmative answer to a question asked by Alex Dugas.

Theorem 6. *Let Λ and Γ be finite dimensional k-algebras whose semisimple quotients are separable. Assume that at least one of them is indecomposable, and that there exists a stable equivalence of Morita type between Λ and Γ. Then the following assertions hold:*
(1) There exists a k-algebra Δ, Morita equivalent to Λ, and an injective ring homomorphism $\Gamma \hookrightarrow \Delta$ such that the restriction and induction functors are a Frobenius pair.

(2) There exists a k-algebra Δ, Morita equivalent to Λ, and an injective ring homomorphism $\Gamma \hookrightarrow \Delta$ such that Δ/Γ is a Frobenius extension.
(3) There exists a k-algebra Δ, Morita equivalent to Λ, and an injective ring homomorphism $\Gamma \hookrightarrow \Delta$ such that

$${}_\Gamma\Delta_\Gamma = {}_\Gamma\Gamma_\Gamma \oplus {}_\Gamma P_\Gamma \quad \textit{and} \quad {}_\Delta\Delta \otimes_\Gamma \Delta_\Delta \simeq {}_\Delta\Delta_\Delta \oplus {}_\Delta Q_\Delta$$

for projective bimodules ${}_\Gamma P_\Gamma$ and ${}_\Delta Q_\Delta$, and Δ is a Frobenius Γ-coring with comultiplication given by the injection of ${}_\Delta\Delta_\Delta$ into ${}_\Delta\Delta \otimes_\Gamma \Delta_\Delta$, and counit given by the projection of ${}_\Gamma\Delta_\Gamma$ onto ${}_\Gamma\Gamma_\Gamma$.

Proof. (1) Suppose ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Delta$ are indecomposable bimodules that induce a stable equivalence of Morita type. Let $\Delta = End_\Lambda(M)$. By the proof of (1)⇒(2) of [4, Corollary 5.1], we have that

$$Res_\Gamma^\Delta \simeq (- \otimes_\Lambda M_\Gamma) \circ Hom_\Delta(M, -)$$

and

$$Ind_\Gamma^\Delta \simeq (- \otimes_\Lambda M_\Delta) \circ (- \otimes_\Gamma N_\Lambda).$$

Now $- \otimes_\Lambda M_\Gamma$ is a right and left adjoint of $- \otimes_\Gamma N_\Lambda$ by [4, Corollary 3.1,(2)], and $Hom_\Delta(M, -)$ is a right and left adjoint of $- \otimes_\Lambda M_\Delta$ because they are inverse equivalences, so Res_Γ^Δ is a right and left adjoint of Ind_Γ^Δ.
(2) follows from (1) and Remark 5.
(3) follows immediately from the above observation that a Frobenius extension is also a Frobenius coring. □

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