INDUCTION FUNCTORS FOR THE DOI-KOPPINEN UNIFIED HOPF MODULES

S. CAENEPEEL
University of Brussels, VUB
Faculty of Applied Sciences,
Pleinlaan 2
B-1050 Brussels, Belgium

and

Ş. RAIANU
University of Bucharest
Faculty of Mathematics
Str. Academiei 14
Bucharest, Romania

Abstract. Let (A, B, D), (A', B', D') be two triples consisting of a Hopf algebra A, an A-comodule algebra B and an A-module coalgebra D. Given $\alpha\, A \to A'$, $\beta\, B \to B'$ and $\delta\, D \to D'$, we define an induction functor between the two corresponding categories of Doi-Koppinen Hopf modules, and we prove that this functor has a right adjoint; this right adjoint is constructed using the cotensor product. We then investigate when this induction functor and its adjoint are inverse equivalences. We find a necessary and sufficient condition, which turns out to be of Galois-type in some special cases. To be able to prove our result, we have to introduce Doi-Koppinen Hopf bimodules and the bitensor product.

Introduction

Recently, Y. Doi [4] and, independently, M. Koppinen [6] constructed a Hopf algebraic generalization of modules graded by a G-set. They considered threetuples (A, B, D) consisting of a Hopf algebra A over a commutative ring k, an A-comodule algebra B and an A-module coalgebra D. An (A, B, D)-Hopf module is then a module which is at once a B-module and a D-comodule, and which satisfies certain compatibility relations (cf. Sec. 1). In the case where A is a groupring, B a G-graded ring and D the grouplike coalgebra kX on a G-set X, these Hopf modules turn out to be exactly the modules graded by the G-set X, which were studied in [8], [10] and [7].

The starting point of this paper was an attempt to extend the results of Menini [7], which extends [11], where all adjoint pairs of functors between categories of graded modules by G-sets are described, to generalized Hopf modules. This would have meant in particular giving generalizations of the induced and the coinduced functors. In this paper we give a generalization of the induction functor and find conditions for this functor to be an equivalence. Thus, the structure of our paper closely resembles the structure of the first two sections in [4].

In Section 1, we introduce the induction functor: given maps $\alpha: A \to A'$, $\beta: B \to B'$, $\delta: D \to D'$, we have a functor $F = \bullet \otimes_B B': \mathcal{M}(A)_B^D \to \mathcal{M}(A')_{B'}^{D'}$. This func-

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tor has a right adjoint G which is constructed in a completely dual way; it therefore involves the so-called cotensor product, which we will denote by $\square_D$. For an (A', B', D')-Hopf module M', $G(M') = M' \square_{D'} D$.

We investigate the following question: when are the functors F and G inverse equivalences ? A necessary condition is easy to find: the k-modules $B\otimes D$ and $D\otimes B$ may be given the structure of (A, B, D)-Hopf module; a necessary condition is of course that $G(F(D \otimes B)) \cong D \otimes B$ and $F(G(B' \otimes D')) \cong B' \otimes D'$. This condition turns out to be very natural: in the case $A = D = k$, $A' = D'$, $\alpha = \delta = \eta_{A'}$, they turn out to be $B = B'^{coA'}$ and B' is A'-Galois in the sense of [4].

In Section 2, we prove that, over a field k, these two conditions are also sufficient in order to have category equivalence, if we furthermore assume that B' is B-flat and that D is D'-coflat. The procedure is the following: we first introduce the notions of Hopf bimodules and bitensor product. The bitensor product over an algebra B and a coalgebra D is denoted by $\otimes_B^D$. The result then follows from the fact that the two functors F and G of Section 1 are obtained by bitensoring up over certain Hopf modules, and from the fact that—under certain flatness conditions—the bitensor product is associative.

1. The induction functor and its adjoint

Notations

Throughout this paper, k will be a unitary commutative ring, and

$$(A, \Delta_A, \varepsilon_A, \mu_A, \eta_A, S_A)$$

will be a k-Hopf algebra. We will assume that the antipode S is bijective. B will be a right A-comodule algebra, with structure map $\rho_B: B \to B \otimes A$. D will be a k-flat right A-module coalgebra, with structure maps $\psi_D: D\otimes A \to D$ and $\Delta_D: D \to D\otimes D$. We will use Sweedler's notations extensively, for example

$$\begin{aligned} \Delta_D(d) &= \sum d_1 \otimes d_2 \\ \rho_B(b) &= \sum b_0 \otimes b_1 \\ \psi(d \otimes a) &= d \leftharpoonup a \end{aligned}$$

for $d \in D$, $b \in B$, $a \in A$. The subscripts A, B, D will be omitted whenever no confusion is possible. A right (A, B, D)-*Hopf module* is a k-module M, which is at once a right D-comodule and a right B-module satisfying

$$\rho_M(mb) = \sum m_0 b_0 \otimes m_1 \leftharpoonup b_1 \tag{1}$$

for all $b \in B$, $m \in M$. $\mathcal{M}(A)_B^D$ will be the category of right (A, B, D)-Hopf modules and B-module D-comodule homomorphisms.

Similarly, a left (A, B, D)-Hopf module is a k-module N, which is a left D-comodule and a left B-module satisfying

$$\rho_N(bn) = \sum (n_{-1} \leftharpoonup S^{-1}(b_1)) \otimes b_0 n_0 \tag{2}$$

for all $b \in B, n \in N$. The corresponding category is denoted by ${}^D_B\mathcal{M}(A)$.

A left-right (A, B, D)-Hopf module is a k-module P, which is a right D-comodule and a left B-module satisfying

$$\rho_P(bp) = \sum b_0 p_0 \otimes (p_1 \leftharpoonup S(b_1)) \tag{3}$$

for all $b \in B$, $p \in P$. The corresponding category is denoted by ${}_B\mathcal{M}(A)^D$. A right-left (A, B, D)-Hopf module is a k-module Q, which is a left D-comodule and a right B-module satisfying

$$\rho_Q(qb) = \sum (q_{-1} \leftharpoonup b_1) \otimes q_0 b_0 \tag{4}$$

for all $b \in B$, $q \in Q$. The corresponding category is denoted by ${}^D\mathcal{M}(A)_B$.

Notice that 16 different versions of this definition are possible: we may consider the four categories mentioned above in each of the following cases: B a left or right A-comodule algebra, and D a left or right A-module coalgebra. These Hopf modules were introduced by Doi (cf. [4]), and, independently, by Koppinen (cf. [6]). They are generalizations of many other notions, as is made clear by the following examples.

Examples

1. An (A, A, A)-Hopf module is an A-Hopf module in the sense of [13].
2. An (A, B, A)-Hopf module is an (A, B)-Hopf module in the sense of [3].
3. An (A, A, D)-Hopf module is a $[D, A]$-Hopf module in the sense of [3].
4. If G is a group, and B is a graded k-algebra, then a (kG, B, kG)-Hopf module is a G-graded B-module.
5. If G is a group, B a G-graded k-algebra, X a right G-set, then a (kG, B, kX)-Hopf module is a module graded by X, cf. [8] and [10] for details.

Theorem 1.1 *Consider two threetuples $\mathcal{A} = (A, B, D)$ and $\mathcal{A}' = (A', B', D')$ as above, and suppose that we have maps $\alpha: A \to A'$, $\beta: B \to B'$ and $\delta: D \to D'$ which are respectively Hopf algebra, algebra and coalgebra maps satisfying*

$$\delta(d \leftharpoonup a) = \delta(d) \leftharpoonup \alpha(a) \tag{5}$$

$$\rho_{B'}(\beta(b)) = \sum \beta(b_0) \otimes \alpha(b_1) \tag{6}$$

for all $a \in A$, $b \in B$, $d \in D$. Then we have a functor $F: \mathcal{M}(A)_B^D \to \mathcal{M}(A')_{B'}^{D'}$, defined as follows: $F(M) = M \otimes_B B' = M'$, where B' is a left B-module via β and with structure maps defined by

$$(m \otimes b')c' = m \otimes b'c'$$

$$\rho_{M'}(m \otimes b') = \sum m_0 \otimes b'_0 \otimes (\delta(m_1) \leftharpoonup b'_1)$$

for all $m \in M$, $b', c' \in B'$. F is called the induction functor from $\mathcal{M}(A)_B^D$ to $\mathcal{M}(A')_{B'}^{D'}$.

Proof. The proof consists of a long, but routine verification. We restrict here to showing that M' satisfies condition (1). Take $n = m \otimes b' \in M'$ and $c' \in B'$. Then

$$\begin{aligned}\sum n_0 c'_0 \otimes n_1 \leftharpoonup c'_1 &= \sum m_0 \otimes b'_0 c'_0 \otimes \delta(m_1) \leftharpoonup (b'_1 c'_1)\\ &= \rho_{M'}(m \otimes b'c')\\ &= \rho_{M'}(n \leftharpoonup c')\end{aligned}$$

□

We will now dualize the construction of Theorem 1.1, and this will provide a right adjoint to the functor F. First recall that the cotensor product $M \,\square_D\, N$ of a right D-comodule M and a left D-comodule N is given by

$$M \,\square_D\, N = \{\sum_i m_i \otimes n_i \in M \otimes N \mid \sum_i m_{i_0} \otimes m_{i_1} \otimes n_i = \sum_i m_i \otimes n_{i_{-1}} \otimes n_{i_0}\},$$

that is, $M \,\square_D\, N$ fits into an exact sequence

$$1 \to M \,\square_D\, N \to M \otimes N \rightrightarrows M \otimes D \otimes N,$$

where the two maps $M \otimes N \to M \otimes D \otimes N$ are $\rho_M \otimes I_N$ and $I_M \otimes \rho_N$.

Lemma 1.2 *If D is flat as a k-module, and M' is a right D'-comodule, then $\rho_M = I_{M'} \otimes \Delta_D$ is a right D-comodule structure map on $M = M' \,\square_{D'}\, D$.*

Proof. We have to show that, for all $\sum_i m'_i \otimes d_i \in M$:

$$x = \rho_M(\sum_i m'_i \otimes d_i) = \sum_i m'_i \otimes d_{i_1} \otimes d_{i_2} \in M \otimes D$$

We have an exact sequence

$$0 \to M \to M' \otimes D \rightrightarrows M' \otimes D' \otimes D$$

Since D is flat, we have another exact sequence

$$0 \to M \otimes D \to M' \otimes D \otimes D \rightrightarrows M' \otimes D' \otimes D \otimes D$$

Therefore, in order to show that $x \in M \otimes D$, it suffices to show that

$$(I_{M'} \otimes \rho_D \otimes I_D)(x) = (\rho_{M'} \otimes I_D \otimes I_D)(x)$$

Indeed, we have

$$\begin{aligned}(\rho_{M'} \otimes I_D \otimes I_D)(x) &= (\rho_{M'} \otimes I_D \otimes I_D)(I_{M'} \otimes \Delta_D)(\sum_i m'_i \otimes d_i)\\ &= (I_{M' \otimes D'} \otimes \Delta_D)(\rho_{M'} \otimes I_D)(\sum_i m'_i \otimes d_i)\end{aligned}$$

$$\begin{aligned}
&= (I_{M'\otimes D'} \otimes \Delta_D)(\sum_i m'_{i_0} \otimes m'_{i_1} \otimes d_i) \\
&= (I_{M'\otimes D'} \otimes \Delta_D)(\sum_i m'_i \otimes \delta(d_{i_1}) \otimes d_{i_2}) \\
&= \sum_i m'_i \otimes \delta(d_{i_1}) \otimes d_{i_2} \otimes d_{i_3} \\
&= (I_{M'} \otimes \rho_D \otimes I_D)(x)
\end{aligned}$$

□

Theorem 1.3 *Under the assumptions of Theorem 1.1, and with D k-flat, we have a functor $G: \mathcal{M}(A')_{B'}^{D'} \to \mathcal{M}(A)_B^D$ which is right adjoint to F. G is defined by*

$$G(M') = M = M' \square_{D'} D$$

for all $M' \in \mathcal{M}(A')_{B'}^{D'}$, and with structure maps

$$\rho_M(\sum_i m'_i \otimes d_i) = \sum_i m'_i \otimes d_{i_1} \otimes d_{i_2} \tag{7}$$

$$(\sum_i m'_i \otimes d_i)b = \sum_i m'_i\beta(b_0) \otimes (d_i \leftharpoonup b_1) \tag{8}$$

for all $b \in B$, $\sum_i m'_i \otimes d_i \in M$.

Proof. Let us first show that M is an object of $\mathcal{M}(A)_B^D$. We have already seen in Lemma 1.2 that M is a right D-comodule. In order to prove that M is a right B-module, we need to show that $\sum_i m'_i \otimes d_i \in M$ implies $(\sum_i m'_i \otimes d_i)b \in M$, for all $b \in B$. On one hand we have

$$(I_{M'} \otimes \rho_D)(\sum_i m'_i\beta(b_0) \otimes (d_i \leftharpoonup b_1)) = \sum_i m'_i\beta(b_0) \otimes \delta(d_{i_1} \leftharpoonup b_1) \otimes d_{i_1} \leftharpoonup b_2 \tag{9}$$

On the other hand

$$(\rho_{M'} \otimes I_D)(\sum_i m'_i\beta(b_0) \otimes (d_i \leftharpoonup b_1)) = \sum_i m'_{i_0}\beta(b_0) \otimes m'_{i_1} \leftharpoonup \alpha(b_1) \otimes d_i \leftharpoonup b_2, \tag{10}$$

since $\rho_{B'}(\beta(b)) = \sum \beta(b_0) \otimes \alpha(b_1)$ for all $b \in B$.

Now let $\sum_b \beta(b_0) \otimes \alpha(b_1) \otimes b_2$ act on the right to the identity

$$\sum_i m'_i \otimes \delta(d_{i_1}) \otimes d_{i_2} = \sum_i m'_{i_0} \otimes m'_{i_1} \otimes d_i$$

This yields the equality of (9) and (10), and this is exactly what we have to show.

In order to show that $M \in \mathcal{M}(A)_B^D$, we still have to verify (1). Take $m = \sum_i m'_i \otimes d_i \in M$. Then

$$\rho_M(mb) = \rho_M(\sum_i m'_i\beta(b_0) \otimes (d_i \leftharpoonup b_1))$$

$$
\begin{aligned}
&= \sum_i m_i'\beta(b_0) \otimes (d_{i_1} \leftharpoonup b_1) \otimes (d_{i_2} \leftharpoonup b_2) \\
&= \sum_i (m_i' \otimes d_{i_1}) b_0 \otimes (d_{i_2} \leftharpoonup b_2) \\
&= \sum m_0 b_0 \otimes (m_1 \leftharpoonup b_1).
\end{aligned}
$$

The functorial properties are straightforward. Let us finally show that G is a right adjoint to F. Take $M \in \underline{\mathcal{C}} = \mathcal{M}(A)_B^D$, $N' \in \underline{\mathcal{C}}' = \mathcal{M}'(A')_{B'}^D$, and put $M' = F(M)$, $N = G(N')$. We define

$$\Phi: \mathrm{Hom}_{\underline{\mathcal{C}}'}(M', N') \to \mathrm{Hom}_{\underline{\mathcal{C}}}(M, N)$$

and

$$\Psi: \mathrm{Hom}_{\underline{\mathcal{C}}}(M, N) \to \mathrm{Hom}_{\underline{\mathcal{C}}'}(M', N')$$

as follows. For $u: M' \to N'$, $v: M \to N$, $m \in M$, $b' \in B'$, we take

$$
\begin{aligned}
\Phi(u)(m) &= \sum u(m_0 \otimes 1) \otimes m_1 \\
\Psi(v)(m \otimes b') &= ((I_{N'} \otimes \varepsilon_D)(v(m)))b'
\end{aligned}
$$

We have to check that $\Phi(u)(m) \in N = N' \,\square_{D'}\, D$. Indeed

$$
\begin{aligned}
(\rho_{N'} \otimes I_D)(\Phi(u)(m)) &= \sum (u \otimes I)(\rho_{M'}(1 \otimes m_0)) \otimes m_1 \\
&= \sum u(1 \otimes m_0) \otimes \delta(m_1) \otimes m_2 \\
&= (I \otimes \rho_D)(\Phi(u)(m)).
\end{aligned}
$$

Φ and Ψ are inverses one to another: for all $m \in M$, $b' \in B'$, we have

$$
\begin{aligned}
\Psi(\Phi(u))(m \otimes b') &= ((I_{N'} \otimes \varepsilon_D)(\Phi(u)(m)))b' \\
&= u(m \otimes 1)b' = u(m \otimes b')
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(\Psi(v))(m) &= \sum \Psi(v)(m_0 \otimes 1) \otimes m_1 \\
&= \sum (I_{N'} \otimes \varepsilon_D) v(m_0) \otimes m_1 \\
&= ((I_{N'} \otimes \varepsilon_D \otimes I_D) \circ (v \otimes I_D) \circ \rho_M)(m) \\
&= ((I_{N'} \otimes \varepsilon_D \otimes I_D) \circ \rho_N \circ v)(m) \\
&= v(m)
\end{aligned}
$$

□

Examples
1) Let (A, B, D) be as before, and consider the threetuple (A, k, D) together with the maps $I_A: A \to A$, $\eta_B: k \to B$, $I_D: D \to D$. Then $\mathcal{M}(A)_k^D = \text{comod-}D$, and the

functor G of Theorem 1.3 is the functor forgetting the B-module structure. Since D itself is a D-comodule, we may consider $F(D) = D \otimes B \in \mathcal{M}(A)_B^D$, with structure maps

$$(d \otimes b)c = d \otimes bc \tag{11}$$
$$\rho_{D\otimes B}(d \otimes b) = \sum d_1 \otimes b_0 \otimes d_2 \leftharpoonup b_1 \tag{12}$$

for all $b, c \in B$, $d \in D$. This Hopf module structure on $D \otimes B$ has allready been considered by Doi [4, ex. 1.4a].

2) In a dual way, consider (A, B, k) and the maps I_A, I_B and $\varepsilon_D: D \to k$. Now $\mathcal{M}(A)_B^k = $ mod-B, and F is the functor forgetting the D-comodule structure. Now B is a right B-module, and $G(B) = B \otimes D \in \mathcal{M}(A)_B^D$, with structure maps

$$(b \otimes d)c = \sum bc_0 \otimes d \leftharpoonup c_1 \tag{13}$$
$$\rho_{B\otimes D}(b \otimes d) = \sum b \otimes d_1 \otimes d_2 \tag{14}$$

Observe that $B \otimes D$ and $D \otimes B$ are isomorphic objects in $\mathcal{M}(A)_B^D$. Indeed, the map

$$\alpha: D \otimes B \to B \otimes D$$

given by

$$\alpha(d \otimes b) = \sum b_0 \otimes d \leftharpoonup b_1 \tag{15}$$

preserves the B-module and D-comodule structure defined above, and has an inverse defined as follows:

$$\alpha^{-1}(b \otimes d) = \sum d \leftharpoonup S^{-1}(b_1) \otimes b_0$$

3) Let G, G' be groups, $f: G \to G'$ a group morphism, X be a right G-set and X' a right G'-set. Suppose that $\Phi: X \to X'$ is such that $\Phi(xg) = \Phi(x)f(g)$ for all $g \in G$ and $x \in X$. Suppose that R is a G-graded ring and that R' is a G'-graded ring, and let $\Psi: R \to R'$ be a ring morphism such that $\Psi(R_g) \subset R'_{f(g)}$ for all $g \in G$. The functor $T^*: (G, A, R)$-$gr \to (G', A', R')$-gr defined in [7, 2.1] is a particular case of our induced functor if we take

$$\begin{array}{lll} A = \mathbb{Z}G & A' = \mathbb{Z}G' & \alpha: A \to A' \text{ induced by } f \\ B = R & B' = R' & \beta = \Psi: B \to B' \\ D = \mathbb{Z}X & D' = \mathbb{Z}X' & \delta: D \to D' \text{ induced by } \Phi \end{array}$$

Thus our induced functor generalizes all induced functors appearing in Graded Ring Theory.

4) Let $A' \subset A$ be a k-direct Hopf subalgebra which is flat over k, and $B' = BJ \otimes_A JA'$. Then our induced functor corresponding to the inclusions $A' \subset A$, $B' \subset B, D' = A' \subset A = D$ is the induced functor for the situation in [12, 3.11(2)]. It unifies the usual induction from invariants and induction from semiinvariants in [2].

5) Doi's induced functor in [4, 1.5] is not a particular case of ours, unless $B_x = C$ (in the notation of [4]), e.g., if $\mathrm{Ann}_{B\otimes A}(x) = 0$. However, $(\bullet)_x \cong (\bullet) \Box_D kx$. We will come back to this example later.

A natural question that arises is the following. Suppose that we are in the situation of Theorem 1.3. When are F and G inverse equivalences ? Or, equivalently, when are the canonical maps

$$\pi_M: M \to G(F(M)) = (M \otimes_B B') \Box_{D'} D$$

$$\phi_{N'}: F(G(N')) = (N' \Box_{D'} D) \otimes_B B' \to N'$$

defined by the adjointness of the functors F and G isomorphisms, for all $M \in \mathcal{M}(A)_B^D$ and $N' \in \mathcal{M}(A')_{B'}^{D'}$? π_M and $\phi_{N'}$ are defined as follows:

$$\pi_M(m) = \sum (m_0 \otimes 1) \otimes m_1 \tag{16}$$

$$\phi_{N'}(\sum_i n_i' \otimes d_i \otimes b') = \sum_i \varepsilon(d_i) n_i' b' \tag{17}$$

for $m \in M$, $b' \in B'$ and $\sum_i n_i' \otimes d_i \in N' \Box_{D'} D$. A necessary condition for all π_M and $\phi_{N'}$ to be isomorphisms is of course that $\pi_{D\otimes B}$ and $\phi_{B'\otimes D'}$ are isomorphisms. This leads to the following definition.

Definition 1.4 *Suppose that $\alpha: A \to A'$, $\beta: B \to B'$ and $\delta: D \to D'$ satisfy conditions (5, 6). Then we say that (α, β, δ) is a Galois threetuple if $\pi_{D\otimes B}$ and $\phi_{B'\otimes D'}$ are isomorphisms.*

Before justifying this definition, let us describe $\pi_{D\otimes B}$ and $\phi_{B'\otimes D'}$ explicitely.

$$G(F(D \otimes B)) = ((D \otimes B) \otimes_B B') \Box_{D'} D \cong (D \otimes B') \Box_{D'} D$$

and

$$\pi_{D\otimes B}(d \otimes b) = \sum d_1 \otimes \beta(b_0) \otimes d_2 \leftharpoonup b_1 \tag{18}$$

for all $b \in B$ and $d \in D$.

Similarly

$$F(G(B' \otimes D')) = ((B' \otimes D') \Box_{D'} D) \otimes_B B' \cong (B' \otimes D) \otimes_B B'$$

and

$$\phi_{B'\otimes D'}(b' \otimes d \otimes c') = \sum b'c_0' \otimes \delta(d) \leftharpoonup c_1' \tag{19}$$

for all $b', c' \in B'$ and $d \in D$.

Examples

1) Take $A = D = k$, $A' = D'$, $\alpha = \delta = \eta_{A'}$. Then $\mathcal{M}(A)_B^D = $ mod-B. For $M' \in \mathcal{M}(A')_{B'}^{D'}$, we have that

$$G(M') = M'^{\mathrm{co}A'} = \{m' \in M' \mid \rho_{M'}(m') = m' \otimes 1\}$$

Now $D \otimes B = B$, $F(B) = B'$, $G(F(B)) = B'^{\mathrm{co}A'}$ and $\pi_{D\otimes B}: B \to B'^{\mathrm{co}A'}$ is given by $\pi_{D\otimes B}(b) = \beta(b)$ (we leave it to the reader to show that $\beta(b) \in B'^{\mathrm{co}A'}$).

Next $B' \otimes D' = B' \otimes A'$, $G(B' \otimes A') = B'$, $F(G(B' \otimes A')) = B' \otimes_B B'$, and

$$\phi_{B'\otimes D'}: B' \otimes_B B' \to B' \otimes A'$$

is given by $\phi_{B'\otimes D'}(b' \otimes c') = \sum b'c'_0 \otimes c'_1$.

It follows that $(\eta_{A'}, \beta, \eta_{A'})$ is Galois if and only if $B \cong B'^{\mathrm{co}A'}$ and $B \to B'$ is an A-Galois extension in the sense of [4, Lemma 2.2].

2) Example 1) can be dualized as follows: take $A = B$, $A' = B' = k$, $\alpha = \beta = \varepsilon_A$. Then $\mathcal{M}(A')^{D'}_{B'} =$ comod-D'. Now for $M \in \mathcal{M}(A)^D_B$, $F(M) = M \otimes_A k = \overline{M}$ is the coequalizer of

$$M \otimes A \rightrightarrows M \to \overline{M} \to 0$$

where the two maps $M \otimes A \rightrightarrows M$ are given by $m \otimes a \to ma$ and $m \otimes a \to \varepsilon(a)m$.

Now $B' \otimes D' = D'$, $G(D') = D$ and $F(G(D')) = \overline{D}$. The map $\phi_{B'\otimes D'}: \overline{D} \to D'$ is given by $\phi_{B'\otimes D'}(\overline{d}) = \delta(d)$ (we leave it to the reader to check that $\phi_{B'\otimes D'}$ is well-defined).

$D \otimes B = D \otimes A$, $F(D \otimes A) = D$ and $G(F(D \otimes A)) = D \,\square_{D'}\, D$. The map

$$\pi_{D\otimes B}: D \otimes B \to D \,\square_{D'}\, D$$

is given by $\pi_{D\otimes B}(d \otimes a) = \sum d_1 \otimes (d_2 \leftharpoonup a)$. Therefore $(\varepsilon_A, \varepsilon_A, \delta)$ is Galois if $\overline{D} = D'$ and $\pi_{D\otimes B}$ is an isomorphism. In this case we might call $D \to \overline{D}$ an *A-Galois coextension.*

2. Equivalences between categories of unified Hopf modules

In this Section, we will show that the functors F and G introduced in Section 1 are inverse equivalences if (α, β, δ) is Galois in the sense of 1.4 and if some additional flatness conditions are satisfied. We will from now on assume that the ground ring k is a field.

Definition 2.1 *Let $\mathcal{A} = (A, B, D)$ and $\mathcal{A}' = (A', B', D')$ be as in Section 1. A k-module Q is then called an $(\mathcal{A}, \mathcal{A}')$-Hopf bimodule if it is a left (A, B, D)-Hopf module and a right (A', B', D')-Hopf module such that the following conditions are satisfied:*

$$\begin{aligned}
(bq)b' &= b(qb') && (20)\\
(I_D \otimes \rho^r_Q) \circ \rho^\ell_Q &= (\rho^\ell_Q \otimes I_{D'}) \circ \rho^r_Q && (21)\\
\rho^r_Q(bq) &= (b \otimes I_{A'})\rho^r_Q(q) && (22)\\
\rho^\ell_Q(qb') &= \rho^\ell_Q(q)(I_A \otimes b') && (23)
\end{aligned}$$

Condition (20) just means that Q is a (B, B')-bimodule, and (21) means that Q is a (D, D')-bicomodule, and justifies the Sweedler notation

$$((I_D \otimes \rho^r_Q) \circ \rho^\ell_Q)(q) = ((\rho^\ell_Q \otimes I_{D'}) \circ \rho^r_Q) = \sum q_{-1} \otimes q_0 \otimes q_1$$

The category of $(\mathcal{A}, \mathcal{A}')$-Hopf bimodules will be denoted by ${}^D_B\mathcal{M}(A, A')^{D'}_{B'}$. In the sequel, we will use Hopf bimodules to define functors between categories of Hopf modules. First, we will introduce a new version of the tensor product.

Definition 2.2 *For $M \in \mathcal{M}(A)_B^D$, $N \in {}_B^D\mathcal{M}(A)$, the bitensor product $M \otimes_B^D N$ is the image of $M \square_D N$ in $M \otimes_B N$.*

We therefore have the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & M \otimes B \otimes N & & \\
 & & & & \downarrow\downarrow & & \\
0 & \longrightarrow & M \square_D N & \longrightarrow & M \otimes N & \rightrightarrows & M \otimes D \otimes N \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_B^D N & \longrightarrow & M \otimes_B N & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

If $A = kG$, with G a group, B is a G-graded ring, and $D = kX$, where X is a G-set, then the bitensor product coincides with Menini's version of the tensor product, cf. [7, 1.1].

Lemma 2.3 *Suppose that k is a field, and let $M \in \mathcal{M}(A)_B^D$, $N \in {}_{B'}^{D'}\mathcal{M}(A')$, $P \in {}_{B''}^{D''}\mathcal{M}(A'', A)_B^D$, $Q \in {}_B^D\mathcal{M}(A, A')_{B'}^{D'}$. Then*

$$M \otimes_B^D Q \in \mathcal{M}(A')_{B'}^{D'} \tag{24}$$

$$Q \otimes_{B'}^{D'} N' \in {}_B^D\mathcal{M}(A) \tag{25}$$

$$P \otimes_B^D Q \in {}_{B''}^{D''}\mathcal{M}(A'', A')_{B'}^{D'} \tag{26}$$

Proof. We will only prove (24). The proof of the other statements is similar. For $x = \sum_i m_i \otimes q_i \in M \otimes_B^D Q$, we define

$$\begin{aligned}
xb' &= \sum_i m_i \otimes q_i b' \\
\rho^r(x) &= \sum_i m_i \otimes q_{i_0} \otimes q_{i_1}
\end{aligned}$$

for all $b' \in B'$. From (20), it follows that xb' is well-defined, and from (22), it follows that $\rho^r(x)$ is well-defined. Also $xb' \in M \otimes_B^D Q$, since

$$\begin{aligned}
&\sum_i m_i \otimes (q_i b')_{-1} \otimes (q_i b')_0 \\
&= \sum_i m_i \otimes (q_{i_{-1}} \otimes q_{i_0})(1_A \otimes b') \qquad \text{(by (23))} \\
&= \sum_i m_{i_0} \otimes m_{i_1} \otimes q_i b'
\end{aligned}$$

Let us next show that $\rho^r(x) \in (M \otimes_B^D Q) \otimes D'$. To this end, it suffices to show that

$$\sum_i m_i \otimes q_{i_0} \otimes q_{i_1} \in M \square_D Q \otimes D'$$

Here we implicitly assume that $(M \square_D Q) \otimes D' \cong M \square_D (Q \otimes D')$; this is correct since we work over a field k, cf. e.g., [12].

We have an exact sequence

$$0 \longrightarrow M \square_D Q \longrightarrow M \otimes Q \rightrightarrows M \otimes D \otimes Q$$

Now D' is k-flat, so the sequence

$$0 \longrightarrow M \square_D Q \otimes D' \longrightarrow M \otimes Q \otimes D' \rightrightarrows M \otimes D \otimes Q \otimes D'$$

is exact. Now

$$\begin{aligned}
&(I_M \otimes \rho^\ell_Q \otimes I_{D'})(\sum_i m_i \otimes q_{i_0} \otimes q_{i_1}) \\
&= I_M \otimes ((\rho^\ell_Q \otimes I_{D'}) \circ \rho^r_Q)(\sum_i m_i \otimes q_i) \\
&= I_M \otimes ((I_D \otimes \rho^r_Q) \circ \rho^\ell_Q)(\sum_i m_i \otimes q_i) \qquad \text{(by (21))} \\
&= (I_M \otimes I_D \otimes \rho^r_Q)(\sum_i m_i \otimes q_{i_{-1}} \otimes q_{i_0}) \\
&= (I_M \otimes I_D \otimes \rho^r_Q)(\sum_i m_{i_0} \otimes m_{i_1} \otimes q_i) \\
&= (I_M \otimes I_Q \otimes I_{D'})(\sum_i m_i \otimes q_{i_0} \otimes q_{i_1})
\end{aligned}$$

and this implies that $\sum_i m_i \otimes q_{i_0} \otimes q_{i_1} \in M \square_D Q \otimes D'$.

To conclude our proof, we still have to verify condition (1):

$$\begin{aligned}
\rho^r(xb') &= \sum_i m_i \otimes (q_i b')_0 \otimes (q_i b')_1 \\
&= \sum_i m_i \otimes q_{i_0} b'_0 \otimes q_{i_1} \leftharpoonup b'_1 \\
&= \sum x_0 b'_0 \otimes x_1 \leftharpoonup b'_1
\end{aligned}$$

□

Let Q be as in Lemma 2.3. Then we have functors

$$\begin{aligned}
\bullet \otimes^D_B Q &: \mathcal{M}(A)^D_B \longrightarrow \mathcal{M}(A')^{D'}_{B'} \\
Q \otimes^{D'}_{B'} \bullet &: {}^{D'}_{B'}\mathcal{M}(A') \longrightarrow {}^D_B\mathcal{M}(A)
\end{aligned}$$

As a first example, we have to go back to Section 1, where we have defined a right (A, B, D)-module structure on $D \otimes B$ (cf. (11, 12)). We now make $D \otimes B$ into an $(\mathcal{A}, \mathcal{A})$-Hopf bimodule as follows:

$$c(d \otimes b) = \sum (d \leftharpoonup S^{-1}(c_1)) \otimes c_0 b \tag{27}$$

$$\rho^\ell_{D\otimes B}(d \otimes b) = \sum d_1 \otimes d_2 \otimes b \tag{28}$$

for all $b, c \in B$, $d \in D$. It is straightforward to show that $D \otimes B$ satisfies conditions (5, 20, 21, 23). Condition (22) is slightly more complicated:

$$\begin{aligned}
\rho^r(c(d \otimes b)) &= \rho^r(\sum (d \leftharpoonup S^{-1}(c_1)) \otimes c_0 b) \\
&= \sum (d \leftharpoonup S^{-1}(c_1))_1 \otimes (c_0 b)_0 \otimes (d \leftharpoonup S^{-1}(c_1))_2 \leftharpoonup (c_0 b)_1 \\
&= \sum (d_1 \leftharpoonup S^{-1}(c_3)) \otimes c_0 b_0 \otimes (d_2 \leftharpoonup S^{-1}(c_2) c_1 b_1) \\
&= \sum (d_1 \leftharpoonup S^{-1}(c_1)) \otimes c_0 b_0 \otimes (d_2 \leftharpoonup b_1) \\
&= \sum c(d_1 \otimes b_0) \otimes d_2 \leftharpoonup b_1 \\
&= (c \otimes 1_{A'}) \rho^r(d \otimes b)
\end{aligned}$$

In a similar may, we make $B \otimes D$ (cf. (13, 14)) into an $(\mathcal{A}, \mathcal{A})$-Hopf bimodule by adding the following structures:

$$c(b \otimes d) = cb \otimes d \tag{29}$$

$$\rho^{\ell}_{B \otimes D}(b \otimes d) = \sum (d_1 \leftharpoonup S^{-1}(b_1)) \otimes b_0 \otimes d_2 \tag{30}$$

for all $b, c \in B$, $d \in D$. We leave it to the reader to show that the map $\alpha: D \otimes B \to B \otimes D$ (cf. (15) is left B-linear and right D-colinear. Hence the Hopf bimodules $B \otimes D$ and $D \otimes B$ are isomorphic.

Lemma 2.4 *Suppose that $M \in \mathcal{M}(A)^D_B$ and $N \in {}^D_B\mathcal{M}(A)$. Then*

$$(B \otimes D) \otimes^D_B N \cong N$$

in ${}^D_B\mathcal{M}(A)$ and

$$M \otimes^D_B (D \otimes B) \cong M$$

in $\mathcal{M}(A)^D_B$.

Proof. $(B \otimes D) \square_D N$ is the submodule of $B \otimes D \otimes N$ generated by elements of the form $\sum b \otimes n_{-1} \otimes n_0$, with $b \in B$, $n \in N$. Define $f: N \longrightarrow (B \otimes D) \otimes^D_B N$ by $f(n) = 1 \otimes n_{-1} \otimes n_0$. We have a well-defined map $g: (B \otimes D) \otimes_B N \longrightarrow N$ given by $g(b \otimes d \otimes n) = \varepsilon(d) bn$ Indeed, for all $c \in B$, we have that

$$\begin{aligned}
g((b \otimes d)c \otimes n) &= g(\sum bc_0 \otimes (d \leftharpoonup c_1) \otimes n) \\
&= \varepsilon(d) bcn = g((b \otimes d) \otimes cn)
\end{aligned}$$

It is straightforward to show that the restriction of g to $(B \otimes D) \otimes^D_B N$ is an inverse of f. The proof of the second statement is similar. $\square$

Recall that a (left) D-comodule M is called (left) D-coflat if the functor

$$\bullet \square_D M : \text{comod-}D \longrightarrow k\text{-mod}$$

is exact. Similarly, we call $M \in {}^D_B\mathcal{M}(A)$ (left) $\mathcal{A}$-biflat if the functor

$$\bullet \otimes^D_B M : \mathcal{M}(A)^D_B \longrightarrow k\text{-mod}$$

is exact. Similar notions apply to right modules.

If M is a right B-module, P a (B,B')-bimodule and Q a left B'-module, then we have a canonical isomorphism $M\otimes_B(P\otimes_{B'}Q)\cong(M\otimes_B P)\otimes_{B'}Q$. Over a field k, a similar property holds for the cotensor product, cf. [14]. We do not have in general a "mixed" associativity for the tensor and the cotensor product. If M is D-coflat or Q is B'-flat, then we have an isomorphism

$$M\,\Box_D\,(P\otimes_{B'}Q)\cong(M\,\Box_D\,P)\otimes_{B'}Q$$

In the next proposition, we generalize this associative property to the bitensor product.

Proposition 2.5 *Let $M\in\mathcal{M}(A)_B^D$, $Q\in{}_{B'}^{D'}\mathcal{M}(A')$ and $P\in{}_B^D\mathcal{M}(A,A')_{B'}^{D'}$. Suppose that M is right $\mathcal{A}$-biflat or Q is left $\mathcal{A}'$-biflat. Then we have a canonical isomorphism*

$$M\otimes_B^D(P\otimes_{B'}^{D'}Q)\cong(M\otimes_B^D P)\otimes_{B'}^{D'}Q$$

Proof. Suppose that Q is left $\mathcal{A}'$-biflat. We will first show that

$$M\otimes(P\,\Box_{D'}\,Q)\cong(M\otimes P)\,\Box_{D'}\,Q \tag{31}$$

This may be seen as follows: the sequence

$$0\longrightarrow P\,\Box_{D'}\,Q\longrightarrow P\otimes Q\rightrightarrows P\otimes D'\otimes Q$$

is exact, hence, since M is k-flat (k is a field), the sequence

$$0\longrightarrow M\otimes(P\,\Box_{D'}\,Q)\longrightarrow M\otimes P\otimes Q\rightrightarrows M\otimes P\otimes D'\otimes Q$$

is exact, proving (31). Next

$$\begin{aligned}M\otimes(P\otimes_{B'}^{D'}Q) &= M\otimes\mathrm{Im}(P\,\Box_{D'}\,Q\to P\otimes_{B'}Q)\\ &= \mathrm{Im}(M\otimes(P\,\Box_{D'}\,Q)\to M\otimes(P\otimes_{B'}Q))\\ &= \mathrm{Im}((M\otimes P)\,\Box_{D'}\,Q\to(M\otimes P)\otimes_{B'}Q)\\ &= (M\otimes P)\otimes_{B'}^{D'}Q\end{aligned}$$

Let us next show that

$$M\otimes_B(P\otimes_{B'}^{D'}Q)=(M\otimes_B P)\otimes_{B'}^{D'}Q \tag{32}$$

Consider the following commutative diagram

$$\begin{array}{ccccccc}(M\otimes B\otimes P)\otimes_{B'}^{D'}Q & \rightrightarrows & (M\otimes P)\otimes_{B'}^{D'}Q & \longrightarrow & (M\otimes_B P)\otimes_{B'}^{D'}Q & \longrightarrow & 0\\ \cong & & \cong & & & & \\ M\otimes B\otimes(P\otimes_{B'}^{D'}Q) & \rightrightarrows & M\otimes(P\otimes_{B'}^{D'}Q) & \longrightarrow & M\otimes_B(P\otimes_{B'}^{D'}Q) & \longrightarrow & 0\end{array}$$

The toprow is exact: this follows from the definition of the tensor product over B and the biflatness of Q. The bottomrow is exact, from the definition of the tensor product over B. Applying the lemma of 5, we obtain (32).

Similarly, an inspection of the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & (M \square_D P) \otimes_{B'}^{D'} Q & \longrightarrow & (M \otimes P) \otimes_{B'}^{D'} Q & \rightrightarrows & (M \otimes D \otimes P) \otimes_{B'}^{D'} Q \\
 & & & & \cong & & \cong \\
0 & \longrightarrow & M \square_D (P \otimes_{B'}^{D'} Q) & \longrightarrow & M \otimes (P \otimes_{B'}^{D'} Q) & \rightrightarrows & M \otimes D \otimes (P \otimes_{B'}^{D'} Q)
\end{array}$$

tells us that

$$(M \square_D P) \otimes_{B'}^{D'} Q \cong M \square_D (P \otimes_{B'}^{D'} Q)$$

Now look at the diagram

$$\begin{array}{ccc}
M \square_D (P \otimes_{B'}^{D'} Q) & \xrightarrow{f} & M \otimes_B (P \otimes_{B'}^{D'} Q) \\
\cong & & \cong \\
(M \square_D P) \otimes_{B'}^{D'} Q & \xrightarrow{g \otimes I_Q} & (M \otimes_B P) \otimes_{B'}^{D'} Q
\end{array}$$

By the definition of the bitensor product, $\mathrm{Im}(f) = M \otimes_B^D (P \otimes_{B'}^{D'} Q)$, and, since Q is $\mathcal{A}'$-biflat, $\mathrm{Im}(g \otimes I_Q) = \mathrm{Im}(g) \otimes_{B'}^{D'} Q = (M \otimes_B^D P) \otimes_{B'}^{D'} Q$, and this implies the result. The proof under the assumption that M is right $\mathcal{A}$-biflat is similar. □

We now return to the situation of Section 1, that is, we consider maps $\alpha: A \to A'$, $\beta: B \to B'$ and $\delta: D \to D'$. We will show that the induction functor F and its adjoint G are both special cases of the bitensor product. To this end, we make $P = D \otimes B'$ into an $(\mathcal{A}, \mathcal{A}')$-Hopf bimodule as follows:

$$\begin{aligned}
c(d \otimes b') &= \sum (d \leftharpoonup S^{-1}(c_1)) \otimes \beta(c_0) b' && (33)\\
\rho^{\ell}_{D\otimes B'}(d \otimes b') &= \sum d_1 \otimes d_2 \otimes b' && (34)\\
(d \otimes b')c' &= d \otimes b'c' && (35)\\
\rho^{r}_{D\otimes B'}(d \otimes b') &= \sum d_1 \otimes b'_0 \otimes (\delta(d_2) \leftharpoonup b'_1) && (36)
\end{aligned}$$

Similarly, $Q = B' \otimes D$ is an $(\mathcal{A}', \mathcal{A})$-Hopf bimodule, with structure maps

$$\begin{aligned}
c(b' \otimes d) &= cb' \otimes d && (37)\\
\rho^{\ell}_{B'\otimes D}(b' \otimes d) &= \sum \delta(d_1) \leftharpoonup S^{-1}(b'_1) \otimes b'_0 \otimes d_2 && (38)\\
(b' \otimes d)b &= \sum b'\beta(b_0) \otimes (d \leftharpoonup b_1) && (39)\\
\rho^{r}_{B'\otimes D}(b' \otimes d) &= \sum b' \otimes d_1 \otimes d_2 && (40)
\end{aligned}$$

for all $b \in B$, $b', c' \in B'$, $d \in D$. We leave it to the reader to show that P and Q satisfy all the conditions of Definition 2.1. Observe also that, as a right (A', B', D')-Hopf module,

$$P = D \otimes B' \cong (D \otimes B) \otimes_B B' = F(D \otimes B)$$

and, as a right (A, B, D)-Hopf module,

$$Q = B' \otimes D \cong (B' \otimes D') \square_{D'} D = G(B' \otimes D')$$

Lemma 2.6 *For all $M \in \mathcal{M}(A)_B^D$ and $M' \in \mathcal{M}(A')_{B'}^{D'}$ we have that*

$$M \otimes_B^D (D \otimes B') \cong M \otimes_B B' = F(M) \tag{41}$$

in $\mathcal{M}(A')_{B'}^{D'}$ and

$$M' \otimes_{B'}^{D'} (B' \otimes D) \cong M' \square_{D'} D = G(M') \tag{42}$$

in $\mathcal{M}(A)_B^D$. Therefore $F = \bullet \otimes_B^D P$ and $G = \bullet \otimes_{B'}^{D'} Q$.

Proof. Observe that $M \square_D (D \otimes B')$ is generated by elements of the form $\sum m_0 \otimes m_1 \otimes b'$, for $m \in M$, $b' \in B'$. Define

$$f: M \otimes_B B' \longrightarrow M \otimes_B^D (D \otimes B')$$

by

$$f(m \otimes b') = \sum m_0 \otimes m_1 \otimes b'$$

It is straightforward to see that f is well-defined: for all $m \in M$, $b' \in B'$ and $c \in C$, we have the following equality in $M \otimes_B (D \otimes B')$:

$$\begin{aligned} f(m \otimes \beta(c)b') &= \sum m_0 \otimes (m_1 \otimes \beta(c)b') \\ &= \sum m_0 \otimes (m_1 \leftharpoonup (c_2 S^{-1}(c_1))) \otimes \beta(c_0)b' \\ &= \sum m_0 \otimes c_0\big((m_1 \leftharpoonup c_1) \otimes b'\big) \\ &= \sum m_0 c_0 \otimes (m_1 \leftharpoonup c_1) \otimes b' \\ &= f(mc \otimes b') \end{aligned}$$

We also have a well-defined map

$$g: M \otimes_B (D \otimes B') \longrightarrow M \otimes_B B'$$

given by $g(m \otimes d \otimes b') = \varepsilon(d)m \otimes b'$. g restricts to

$$g: M \otimes_B^D (D \otimes B') \longrightarrow M \otimes_B B'$$

and we have that

$$g(\sum m_0 \otimes m_1 \otimes b') = m \otimes b'$$

It is therefore clear that f and g are each others inverses. Also f is a morphism in $\mathcal{M}(A')_{B'}^{D'}$:

$$f(m \otimes b'c') = \sum m_0 \otimes m_1 \otimes b'c' = (\sum m_0 \otimes m_1 \otimes b')c'$$

and

$$\begin{aligned} (f \otimes I_{D'})(\rho(m \otimes b')) &= (f \otimes I_{D'})(\sum m_0 \otimes b'_0 \otimes (\delta(m_1) \leftharpoonup b'_1)) \\ &= \sum m_0 \otimes m_1 \otimes b'_0 \otimes \delta(m_2) \leftharpoonup b'_1 \\ &= \rho(\sum m_0 \otimes m_1 \otimes b') \\ &= \rho(f(m \otimes b')) \end{aligned}$$

and this proves (41).

It is straightforward to see that the (well-defined) map

$$f: M' \otimes_{B'} (B' \otimes D) \longrightarrow M' \otimes D$$

given by $f(m' \otimes b' \otimes d) = m'b' \otimes d$ restricts to a map

$$f: M' \otimes_{B'}^{D'} (B' \otimes D) \longrightarrow M' \square_{D'} D$$

Indeed, if $\sum_i m'_i \otimes b'_i \otimes d_i \in M' \square_{D'} (B' \otimes D)$, then

$$\begin{aligned}&\sum_i m'_{i_0} \otimes m'_{i_1} \otimes b'_i \otimes d_i \\ &= \sum_i m'_i \otimes \delta(d_{i_1}) \leftharpoonup S^{-1}(b'_{i_1}) \otimes b'_{i_0} \otimes d_{i_2}\end{aligned}$$

Apply $I_{M'} \otimes I_{D'} \otimes \rho_{B'} \otimes I_{D'}$ to both sides, to obtain

$$\begin{aligned}&\sum_i m'_{i_0} \otimes m'_{i_1} \otimes b'_{i_0} \otimes b'_{i_1} \otimes d_i \\ &= \sum_i m'_i \otimes \delta(d_{i_1}) \leftharpoonup S^{-1}(b'_{i_2}) \otimes b'_{i_0} \otimes b'_{i_1} \otimes d_{i_2}\end{aligned}$$

Now let the third factor act on the first one, and the fourth one on the second one. We now obtain

$$\begin{aligned}&\sum_i m'_{i_0} b'_{i_0} \otimes m'_{i_1} \leftharpoonup b'_{i_1} \otimes d_i \\ &= \sum_i m'_i b'_{i_0} \otimes \delta(d_{i_1}) \leftharpoonup (S^{-1}(b'_{i_2}) b'_{i_1}) \otimes d_{i_2} \\ &= \sum_i m'_i b'_i \otimes \delta(d_{i_1}) \otimes d_{i_2}\end{aligned}$$

which means exactly that $\sum m'_i b'_i \otimes d_i \in M' \square_{D'} D$.

Define $g: M' \square_{D'} D \longrightarrow M' \otimes_{B'}^{D'} (B' \otimes D)$ by

$$g(\sum_i m'_i \otimes d_i) = \sum_i m'_i \otimes 1 \otimes d_i$$

It is then clear that $f \circ g = I_{M' \square_{D'} D}$, and, for all $\sum_i m'_i \otimes b_i \otimes d_i \in M' \otimes_{B'}^{D'} (B' \otimes D)$:

$$\begin{aligned}(g \circ f)(\sum_i m'_i \otimes b_i \otimes d_i) &= \sum_i m'_i b_i \otimes 1 \otimes d_i \\ &= \sum_i m'_i \otimes b_i \otimes d_i\end{aligned}$$

in $M' \otimes_{B'}^{D'} (B' \otimes D)$. It is straightforward to show that g is B-linear and D-colinear, and this finishes the proof of (42). □

We now come to our main result

Proposition 2.7 *Suppose that k is a field, and let $\mathcal{A} = (A, B, D)$, $\mathcal{A}' = (A', B', D')$, $\alpha: A \to A'$, $\beta: B \to B'$ and $\delta: D \to D'$ be as in Theorems 1.1 and 1.3.*

1. *If B' is left B-flat, and $\phi_{B'\otimes D'}$ is an isomorphism, then $F(G(N')) \cong N'$ for all $N' \in \mathcal{M}(A')_{B'}^{D'}$.*
2. *If D is left D'-coflat, and $\pi_{D\otimes B}$ is an isomorphism, then $G(F(M)) \cong M$ for all $M \in \mathcal{M}(A)_B^D$.*

Proof. We will prove 2). The proof of 1) is similar. From Lemma 2.6, it follows that $P = D \otimes B'$ is left $\mathcal{A}$-biflat (since $(\bullet \otimes_B^D (D \otimes B') = \bullet \otimes_B B' = F)$. Therefore, applying 2.5 and 2.6, we obtain

$$G(F(M)) = \big(M \otimes_B^D (D \otimes B')\big) \otimes_{B'}^{D'} (B' \otimes D) \cong M \otimes_B^D \big((D \otimes B') \otimes_{B'}^{D'} (B' \otimes D)\big)$$

Now the map $\pi_{D\otimes B}: D\otimes B \longrightarrow G(F(D\otimes B)) = (D\otimes B')\Box_{D'} D = (D\otimes B')\otimes_{B'}^{D'}(B'\otimes D)$ is an isomorphism, by assumption. A direct computation shows that $\pi_{D\otimes B}$ is an isomorhism of (A, B, D)-Hopf bimodules. Therefore

$$G(F(M)) = M \otimes_B^D (D \otimes B) \cong M$$

by Lemma 2.4. In a similar way, we may prove that $F(G(N')) \cong N'$ for all $N' \in \mathcal{M}(A')_{B'}^{D'}$.
□

Theorem 2.8 *Under the conditions of Proposition 2.7, the following assertions are equivalent:*

1. *F and G are inverse equivalences;*
2. *(α, β, δ) is Galois (in the sense of 1.4), B' is flat as a left B-module and D is coflat as a left D'-comodule;*
3. *(α, β, δ) is Galois and B' is faithfully flat as a left B-module;*
4. *(α, β, δ) is Galois and D is faithfully coflat as a left D'-comodule.*

Proof. It is clear that condition 1) implies the three other conditions. 2) $\Rightarrow$ 1) follows immediately from Proposition 2.7. Let us show that 3) implies 2). From Proposition 2.7, it follows that $F(G(N')) \cong N'$, for all $N' \in \mathcal{M}(A')_{B'}^{D'}$. Therefore, if

$$0 \longrightarrow N_1' \longrightarrow N_2' \longrightarrow N_3'$$

is exact in $\mathcal{M}(A')_{B'}^{D'}$, then

$$0 \longrightarrow F(G(N_1')) \longrightarrow F(G(N_2')) \longrightarrow F(G(N_3'))$$

is exact, and therefore

$$0 \longrightarrow G(N_1') \longrightarrow G(N_2') \longrightarrow G(N_3')$$

is exact, since F is faithfully exact. It follows that D is D'-coflat. The assertion 4) $\Rightarrow$ 2) may be proved in a similar way.
□

EXAMPLES

1) Let us reconsider the final example of Section 1: Take $A = D = k$, $A' = D'$ and $\alpha = \delta = \eta_{A'}$. Theorem 2.8 now gives the following classical result. Suppose that B' is an A'-comodule algebra, and let $B = B'^{\mathrm{co}A'}$. If B' is faithfully flat as a B-module and an A-Galois extension, then the categories $\mathcal{M}(A')_{B'}^{D'}$ and mod-B are equivalent.

2) In the dual situation, Theorem 2.8 takes the following form: suppose that D is an A-module coalgebra. Write $\overline{D} = D'$, and assume that D is faithfully coflat as a left D'-comodule. If furthermore $D \to \overline{D}$ is an A-Galois coextension, then the categories $\mathcal{M}(A)_B^D$ and comod-$\overline{D}$ are equivalent.

After having seen the bitensor product, the reader might wonder if we can develop a Morita-type theory for the bitensor product. At first glance, it seems unlikely that this would work. Indeed, such a theory would have to agree with the classical theory for modules (cf. [1]), as well as with the one for comodules (cf. [14]). But in the two cases, the arrows have opposite direction! As far as the "adjoint functor" part of the theorem is concerned, a possible answer is the following. Consider a Morita context: In order to have a pair of adjoint functors associated to it, we need that one of the two defining maps is surjective (and therefore injective, cf. [1]). A careful inspection of the proof yields that, in order to have a pair of adjoint functors, it suffices to reverse the direction of one of the two arrows in the definition of Morita context. We then do not need any of the maps to be injective or surjective. A similar observation holds for Morita-Takeuchi contexts, and this restores the symmetry between the two situations. The next theorem is the generalization of the above observations to the bitensor product for generalized Hopf bimodules:

Theorem 2.9 *Suppose that k is a field and that $\mathcal{A} = (A, B, D)$ and $\mathcal{A}' = (A', B', D')$ are as before. Suppose that P is a left biflat $(\mathcal{A}, \mathcal{A}')$-Hopf bimodule and that Q is a left biflat $\mathcal{A}', \mathcal{A}$-Hopf bimodule. Assume furthermore that we are given some Hopf bimodule maps*

$$\pi: B \otimes D \longrightarrow P \otimes_{B'}^{D'} Q$$

and

$$\phi: Q \otimes_B^D P \longrightarrow B' \otimes D'$$

such that the following two diagrams commute:

$$\begin{array}{ccc} P \otimes_{B'}^{D'} Q \otimes_B^D P & \xrightarrow{I_P \otimes \phi} & P \otimes_{B'}^{D'} (B' \otimes D') \\ \uparrow{\scriptstyle \pi \otimes I_P} & & \cong \\ (B \otimes D) \otimes_B^D P & \cong & P \end{array}$$

$$\begin{array}{ccc} Q \otimes_B^D P \otimes_{B'}^{D'} Q & \xrightarrow{\phi \otimes I_Q} & (B' \otimes D') \otimes_{B'}^{D'} Q \\ \uparrow{\scriptstyle I_Q \otimes \pi} & & \cong \\ Q \otimes_B^D (B \otimes D) & \cong & Q \end{array} \qquad (43)$$

Then $G = \bullet \otimes_{B'}^{D'} Q$ is a right adjoint to $F = \bullet \otimes_B^D P$. If π and ϕ are isomorphisms, then F and G are inverse equivalences.

Proof. We will implicitly assume that the bitensor product is associative. Everywhere we do this, it will follow from the biflatness of P or Q. Take $M \in \underline{\mathcal{C}} = \mathcal{M}(A)_B^D$ and $N' \in \underline{\mathcal{C}}' = \mathcal{M}(A')_{B'}^{D'}$. We define

$$\Phi: \mathrm{Hom}_{\underline{\mathcal{C}}'}(M \otimes_B^D P, N') \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}}(M, N' \otimes_{B'}^{D'} Q)$$

and

$$\Psi: \mathrm{Hom}_{\underline{\mathcal{C}}}(M, N' \otimes_{B'}^{D'} Q) \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}'}(M \otimes_B^D P, N')$$

as follows: for $u \in \mathrm{Hom}_{\underline{\mathcal{C}}'}(M \otimes_B^D P, N')$, $\Phi(u)$ is the composition

$$\begin{aligned} M \cong M \otimes_B^D (B \otimes D) &\xrightarrow{I_M \otimes \pi} M \otimes_B^D P \otimes_{B'}^{D'} Q \\ &\xrightarrow{u \otimes I_Q} N' \otimes_{B'}^{D'} Q \end{aligned}$$

For $v \in \mathrm{Hom}_{\underline{\mathcal{C}}}(M, N' \otimes_{B'}^{D'} Q)$, $\Psi(v)$ is the composition

$$\begin{aligned} M \otimes_B^D P &\xrightarrow{v \otimes I_P} N' \otimes_{B'}^{D'} Q \otimes_B^D P \\ &\xrightarrow{I_{N'} \otimes \phi} N' \otimes_{B'}^{D'} (B' \otimes D') \cong N' \end{aligned}$$

Let us show that $(\Psi \circ \Phi)(u) = u$. $(\Psi \circ \Phi)(u)$ is given by the following composition:

$$\begin{aligned} M \otimes_B^D P &\cong M \otimes_B^D (B \otimes D) \otimes_B^D P \\ &\xrightarrow{I_M \otimes \pi \otimes I_P} M \otimes_B^D P \otimes_{B'}^{D'} Q \otimes_B^D P \\ &\xrightarrow{u \otimes I_Q \otimes I_P} N' \otimes_{B'}^{D'} Q \otimes_B^D P \\ &\xrightarrow{I_{N'} \otimes \phi} N' \otimes_{B'}^{D'} (B' \otimes D') \cong N' \end{aligned}$$

Following Sweedler's tradition, we introduce the notation

$$\pi(1_B \otimes d) = \sum d^{(1)} \otimes d^{(2)} \in P \otimes_B Q$$

for $d \in D$. The commutativity of the first diagram in (43) can now be restated as follows:

$$\sum p_{-1}^{(1)} \otimes \phi(p_{-1}^{(2)} \otimes p_0) = \sum p_0 \otimes 1_{B'} \otimes p_1 \tag{44}$$

for all $p \in P$. Now take $\sum_i m_i \otimes p_i \in M \otimes_B^D P$. The image of $\sum_i m_i \otimes p_i$ in $M \otimes_B^D (B \otimes D) \otimes_B^D P$ is $\sum_i m_i \otimes (1 \otimes p_{i_{-1}}) \otimes p_{i_0}$. Now

$$\begin{aligned} &(I_{N'} \otimes \phi)(u \otimes I_Q \otimes I_P)(I_M \otimes \pi \otimes I_P)(\sum_i m_i \otimes (1 \otimes p_{i_{-1}}) \otimes p_{i_0}) \\ &= (I_{N'} \otimes \phi)(u \otimes I_Q \otimes I_P)(\sum_i m_i \otimes p_{i_{-1}}^{(1)} \otimes p_{i_{-1}}^{(2)} \otimes p_{i_0}) \\ &= (I_{N'} \otimes \phi)\big(u(m_i \otimes p_{i_{-1}}^{(1)}) \otimes p_{i_{-1}}^{(2)} \otimes p_{i_0}\big) \\ &= u(\sum_i m_i \otimes p_{i_0}) \otimes 1_{B'} \otimes p_{i_1} \end{aligned}$$

by (44). The image in N' is $\sum_i u(m_i \otimes p_{i_0})\varepsilon(p_{i_1}) = u(\sum_i m_i \otimes p_i)$. Therefore $(\Psi \circ \Phi)(u) = u$. A similar computation shows that $(\Phi \circ \Psi)(v) = v$.

The second assertion is left as an exercise to the reader. □

Examples
As we have remarked in Section 1, Doi's induced functors [4, 1.5] is not a particular case of our functors F and G. Let us show that Doi's functor is obtained by bitensoring up over a Hopf bimodule; thus Doi's functor appears as a special case of 2.9.

Let (A, B, D) be as before, and let x be a grouplike element of D. It is straightforward to check that the map $\pi_x: A \to D$ defined by $\pi_x(a) = x \leftharpoonup a$ is an A-module coalgebra map. Also

$$B_x = \{b \in B \mid \sum b_0 \otimes \pi_x(b_1) = b \otimes x\}$$

is a subalgebra of B, and kx is a subcoalgebra of D. It is clear that

$$\mathcal{M}(k)_{B_x}^{kx} \cong \text{mod-}B_x.$$

We make $kx \otimes B$ into an object of ${}_{B_x}^{kx}\mathcal{M}(k, A)_B^D$ as follows:

$$\begin{aligned}
\rho^r(x \otimes b) &= \sum x \otimes b_0 \otimes \pi_x(b_1) \\
(x \otimes b)c &= x \otimes bc \\
f(x \otimes b) &= x \otimes fb \\
\rho^\ell(x \otimes b) &= x \otimes x \otimes b
\end{aligned}$$

for all $b, c \in B$, $f \in B_x$. We leave it as an exercise to the reader to verify that $kx \otimes B$ meets all necessary requirements. In a similar way, $B \otimes kx \in {}_B^D\mathcal{M}(A, k)_{B_x}^{kx}$, with

$$\begin{aligned}
\rho^r(b \otimes x) &= b \otimes x \otimes x \\
(b \otimes x)f &= bf \otimes x \\
\rho^\ell(b \otimes x) &= x \otimes b \otimes x \\
c(b \otimes x) &= cb \otimes x
\end{aligned}$$

Take $V \in \text{mod-}B_x = \mathcal{M}(k)_{B_x}^{kx}$. Then

$$V \otimes_{B_x}^{kx} (kx \otimes B) \cong V \otimes_{B_x} B \in \mathcal{M}(A)_B^D$$

with structure induced by the right structure of B, that is

$$\begin{aligned}
\rho(v \otimes b) &= \sum v \otimes b_0 \otimes \pi_x(b_1) \\
(v \otimes b)c &= v \otimes bc
\end{aligned}$$

as in [4]. Similarly, for $M \in \mathcal{M}(A)_B^D$, we have

$$\begin{aligned}
M \otimes_B^D (B \otimes kx) &\cong M \,\square_D\, kx \\
&= \{m \in M \mid \rho(m) = m \otimes x\} = M_x
\end{aligned}$$

in the notations of [4]. The induced structure on M_x is given by $\rho(m) = m \otimes x$ and $mf = mf$, for all $m \in M$, $f \in B_x$.

Observe finally that the maps π and ϕ of Theorem 2.9 are the following:

$$\pi: B_x \otimes kx \xrightarrow{\cong} B_x \cong (kx \otimes B) \otimes_B^D (B \otimes kx)$$

and

$$\phi: (B \otimes kx) \otimes_{B_x}^{kx} (kx \otimes B) = B \otimes_{B_x} B \longrightarrow B \otimes D$$

given by $\phi(b' \otimes b) = \sum b' b_0 \otimes (x \leftharpoonup b_1)$, for all $b, b' \in B$.

In [4, th. 2.3], a sufficient condition is given for the above functors to be inverse equivalences. In a similar way, Doi's dual induction functor $(\bullet)^\alpha$ (cf. [4, 1.9]) appears as a special case of Theorem 2.9.

Remark

Since the object $B \otimes D$ plays a special role in Theorem 2.9, we give some more remarks about this object. Assume that k is a field, and let $B\#D^*$ denote the smash product of [4]. An object of ${}_B\mathcal{M}(A)^D$ is a left $B\#D^*$-module, as shown in [4]. It is easy to see that a $B\#D^*$-module which is rational as a D^*-module is an object of ${}_B\mathcal{M}(A)^D$. So ${}_B\mathcal{M}(A)^D$ is a Grothendieck category. Now it was observed in [9] that the subcategory of rational D^*-modules of D^*-mod is in fact equal to $\sigma[_{D^*}D]$, the subcategory of D^*-modules subgenerated by D, i.e. submodules of quotients of direct sums of copies of D. This is implicit in [5], where it is shown that every comodule is a D^*-submodule of $D^{(I)}$ for some set I. Hence the rational D^*-modules are contained in $\sigma[_{D^*}D]$ and since the latter is the smallest closed subcategory of D^*-mod containing D, we have equality. A family of generators for $\sigma[_{D^*}D]$ are D^*/I (I is a left ideal) such that $D^*/I \in \sigma[_{D^*}D]$. Thus a generator for $\sigma[_{D^*}D]$ is $\oplus\{D^*/I \mid D^*/I$ is rational$\}$. We have a similar result for ${}_B\mathcal{M}(A)^D$. Let $M \in {}_B\mathcal{M}(A)^D$. As a D-comodule M may be embedded into $D^{(I)}$, so $B \otimes M$ may be embedded into $B \otimes D^{(I)} \cong (B \otimes D)^{(I)}$, thus $B \otimes M \in \sigma[_{B\#D^*}B \otimes D]$. Thus ${}_B\mathcal{M}(A)^D$ considered as a subcategory of $B\#D^*$-mod is exactly $\sigma[_{B\#D^*}B \otimes D]$. Now consider the forgetful functor

$$F\ {}_B\mathcal{M}(A)^D \longrightarrow \text{comod-}D$$

and its left adjoint $T = B \otimes \bullet$. Being a left adjoint, T commutes with direct sums. Moreover the canonical morphism $(T \circ F)(M) \to M$ is surjective. It follows that T sends generators to generators. Indeed, if U is a generator for comod-D and $M \in {}_B\mathcal{M}(A)^D$, then $U^{(I)} \to F(M) \to 0$, so $T(U^{(I)}) \cong T(U)^{(I)} \to T(F(M)) \to 0$ and $T(F(M)) \to M \to 0$, so M is an epimorphic image of a direct sum of copies of $T(U)$. In conclusion, a generator for ${}_B\mathcal{M}(A)^D$ is $B \otimes \big(\oplus\{D^*/I \mid D^*/I \text{ is rational}\}\big)$, where D^*/I are considered as D-comodules.

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