JOURNAL OF ALGEBRA 165, 204–222 (1994)

Induction and Coinduction for Hopf Algebras: Applications

STEFAAN CAENEPEEL

Faculty of Applied Sciences, Free University of Brussels (VUB), Pleinlaan 2, B-1050 Brussel, Belgium

ŞERBAN RAIANU

Faculty of Mathematics, University of Bucharest, Str. Academiei 14, RO-70109 Bucharest, Romania

AND

FREDDY VAN OYSTAEYEN

Department of Mathematics, University of Antwerp (UIA), Universiteitsplein 1, B-2610 Wilrijk, Belgium

Communicated by Susan Montgomery

Received July 23, 1991

INTRODUCTION

In this paper, we study induction and coinduction from invariants (Section 2) or semivariants (Section 3) for an algebra A that is an H-module algebra for a Hopf algebra H. We restrict our attention to finite dimensional Hopf algebras and in fact usually even to the semisimple case. When A is an H-module algebra we let A^H denote the subalgebra of H-invariants and $A \# H$ is the smash product. In Section 2, the induced (Ind) and coinduced (Coind) functors from A^H-mod to $A \# H$-mod are studied. The main result in Section 2, Theorem 2.1, provides an analogue of Theorem 1.1 of [N] stating that these functors appear in adjoint pairs. In a particular case such a result was known for the induced functor, cf. [D2, DT]. We include some applications of this result. The coinduced functor is then used for describing injective envelopes in $A \# H$-mod and for establishing links between the endomorphism ring of a simple $A \# H$-module and the endomorphism ring of the homogeneous components of its semi-invariant subspace.

0021-8693/94 $6.00

The consideration of semi-invariants is the topic of Section 3. Here we connect Hopf algebra results to results and techniques from graded ring theory. First we have to rework in Theorem 3.1 the statements of Theorem 2.1 for induction and coinduction from semi-invariants.

1. Preliminaries

Let H be a Hopf algebra over the field k, and A an H-module algebra. The notation is that of [S]. For a summary of basic properties, see [CF1]. S will denote the antipode of H. Its composition inverse (when it exists) will be denoted by $\bar{S}$. The comultiplication map is Δ and the augmentation map ε. We denote for each $h \in H$

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

If R is a ring, then R-mod will denote the category of left R-modules. If $R = \bigoplus_{\sigma \in G} R_\sigma$ is a G-graded ring (G is a group), then R-gr will denote the category of left graded R-modules. If $A \# H$ denotes the usual smash product then we have the following:

1.1. *Remark.* A k-vector space M is an $A \# H$-module if and only if M is a left H-module and a left A-module such that

(1.2) $\quad h(am) = \sum_{(h)} (h_{(1)}a)(h_{(2)}m), \qquad$ for $\quad h \in H, a \in A, m \in M.$

Assertions (i) and (iii) of the following result are Lemma 1 of [C]. The proof of (ii) and (iv) is similar.

1.3. Lemma. *Let H be a Hopf algebra, A an H-module algebra, and $M \in A \# H$-mod. For any $a, b \in A$, $h \in H$, $m \in M$, we have:*

(i) $(ha)b = \sum_{(h)} h_{(1)}[a(S(h_{(2)})b)]$;

(ii) $(ha)m = \sum_{(h)} h_{(1)}[a(S(h_{(2)})m)]$.

If S is bijective, then

(iii) $a(hb) = \sum_{(h)} h_{(2)}[(\bar{S}(h_{(1)})a)b]$;

(iv) $a(hm) = \sum_{(h)} h_{(2)}[(\bar{S}(h_{(1)})a)m]$.

If H is finite dimensional, then $G = \mathrm{Alg}(H, k)$, the set of grouplike elements of $H^* = \mathrm{Hom}_k(H, k)$, is a finite group under the convolution product. For $\sigma \in G$, we denote by $\tilde{\sigma}$ the convolution inverse of σ. The unit element of G is ε, the augmentation map. If A is an H-module algebra and $M \in A \# H$-mod, we put for each $\lambda \in G$,

$$A_\lambda = \{a \in A : ha = \lambda(h)a, \text{ for all } h \in H\},$$

and, similarly

$$M_\lambda = \{m \in M : hm = \lambda(h)m, \text{ for all } h \in H\}.$$

Note that A_ε (resp. M_ε) is the subalgebra (resp. subspace) of invariants, usually denoted by A^H (resp. M^H). The set $\bigcup_{\lambda \in G} A_\lambda$ (resp. $\bigcup_{\lambda \in G} M_\lambda$) is called the set of *H-semi-invariants* of A (resp. M).

The following is Lemma 3.9 of [BCM], with the obvious analogues for $A \# H$-modules added:

LEMMA. (i) *If* $\lambda, \mu \in G$, *then* $A_\lambda A_\mu \subset A_{\lambda\mu}$ *and* $A_\lambda M_\mu \subset M_{\lambda\mu}$.

(ii) *The sums* $\sum_{\lambda \in G} A_\lambda$ *and* $\sum_{\lambda \in G} M_\lambda$ *are direct.*

Thus, $S_A = \sum_{\lambda \in G} A_\lambda$ is a G-graded ring, called the *semi-invariant subalgebra* of A, and $S_M = \sum_{\lambda \in G} M_\lambda$ is a graded S_A-module, called the *semi-invariant subspace* of M. Since ε is the unit element of G, M_λ is a left A^H-module for each $\lambda \in G$. It is obvious that if M and M' are $A \# H$-modules, and $f: M \to M'$ is $A \# H$-linear, then $f(M_\lambda) \subset M'_\lambda$, for all $\lambda \in G$. Thus we have a functor $S_{(-)}: A \# H\text{-mod} \to S_A\text{-gr}$, sending M to S_M; obviously, $S_{(-)}$ is a left exact functor. For each $\sigma \in G$, we also have a left exact functor $(-)_\sigma: A \# H\text{-mod} \to A^H\text{-mod}$, sending M to M_σ.

We now define the suspension functors as in the graded case. Let $\sigma \in G$, and define

$$T_\sigma : A \# H\text{-mod} \to A \# H\text{-mod}$$

in the following way: for $M \in A \# H$-mod, let $T_\sigma(M) = M$ as A-modules, and with H-module structure given by $h \cdot m = \sum_{(h)} \bar{\sigma}(h_{(2)})\, h_{(1)} m$. It is straightforward to check that T_σ is a functor. We have the following:

1.5. LEMMA. (i) $T_\sigma \circ T_\tau = T_{\sigma\tau}$;

(ii) $T_\varepsilon = 1_{A \# H\text{-mod}}$;

(iii) T_σ *is an isomorphism with inverse* $T_{\bar{\sigma}}$.

Proof. (i) If $M \in A \# H$-mod, the H-module structure on $T_\sigma(T_\tau(M))$ is given by

$$\begin{aligned} h \cdot m &= \textstyle\sum_{(h)} \bar{\sigma}(h_{(2)})\, h_{(1)} \cdot m = \sum_{(h)} \bar{\sigma}(h_{(3)})\, \bar{\tau}(h_{(2)})\, h_{(1)} m \\ &= \textstyle\sum_{(h)} (\bar{\tau}\bar{\sigma})(h_{(2)})\, h_{(1)} m = \sum_{(h)} (\overline{\sigma\tau})(h_{(2)})\, h_{(1)} m, \end{aligned}$$

which is the H-module structure on $T_{\sigma\tau}(M)$.

Part (ii) is obvious and (iii) follows from (i) and (ii).

Now for each $\lambda \in G$, we define $\Phi_\lambda: H \to H$ by $\Phi_\lambda(h) = \sum_{(h)} \bar{\lambda}(h_{(1)})\, h_{(2)}$, for all $h \in H$. By Lemma 3.11 of [BCM], we have that:

I. $\Phi_\lambda(hl) = \Phi_\lambda(h)\, \Phi_\lambda(l)$, for all $h, l \in H$;

II. $\Phi_\lambda(1) = 1$;

(III) $\Phi_{\lambda\mu}(h) = \Phi_\lambda(\Phi_\mu(h))$, for all $h \in H$;

(IV) $\Phi_\varepsilon(h) = h$, for all $h \in H$.

We can also define $\Psi_\lambda: H \to H$ by $\Psi_\lambda(h) = \sum_{(h)} h_{(1)} \bar{\lambda}(h_{(2)})$, for all $h \in H$. It is obvious that it has properties (I)–(IV) too (only in (III), $\Psi_{\lambda\mu}(h) = \Psi_\mu(\Psi_\lambda(h))$).

We will say that $x \in H$ is a left (resp. right) λ-integral ($\lambda \in G$) if $hx = \lambda(h)x$ (resp. $xh = \lambda(h)x$) for all $h \in H$. Then an ε-integral is what is usually called an integral. The following will be useful in Section 3.

1.6. Lemma. *If H is finite dimensional and semisimple, let $t \in H$ denote the left and right integral of H such that $\varepsilon(t) = 1$ and $S(t) = t$ (see [S, p. 103]). Then for each $\lambda \in G$, the elements $e_\lambda = \Phi_\lambda(t) = \Sigma_{(t)} \bar{\lambda}(t_{(1)})\, t_{(2)}$ and $t_\lambda = \Psi_\lambda(t) = \Sigma_{(t)} t_{(1)} \bar{\lambda}(t_{(2)})$ have the following properties:*

(i) e_λ and t_λ are left and right λ-integrals such that $\lambda(e_\lambda) = \lambda(t_\lambda) = 1$; in fact $e_\lambda = t_\lambda$ for all $\lambda \in G$. However, we will preserve the two different notations in order to allow the reader to recognize which one of the two formulas was used;

(ii) if $\mu \in G$, then $\Phi_\mu(e_\lambda) = e_{\mu\lambda}$ and $\Psi_\mu(t_\lambda) = t_{\lambda\mu}$;

(iii) $S(e_\lambda) = t_{\bar{\lambda}}$ and $S(t_\lambda) = e_{\bar{\lambda}}$.

Proof. (i) Let $h \in H$. Then

$$\begin{aligned} he_\lambda = h\Phi_\lambda(t) &= \Phi_\varepsilon(h)\, \Phi_\lambda(t) = \Phi_\lambda(\Phi_{\bar{\lambda}}(h))\, \Phi_\lambda(t) = \Phi_\lambda(\Phi_{\bar{\lambda}}(h)\, t) \\ &= \Phi_\lambda((\textstyle\sum_{(h)} \lambda(h_{(1)})\, h_{(2)})\, t) = \Phi_\lambda(\varepsilon(\Sigma_{(h)} \lambda(h_{(1)})\, h_{(2)})\, t) \\ &= \Phi_\lambda((\textstyle\sum_{(h)} \lambda(h_{(1)})\, \varepsilon(h_{(2)}))\, t) = \Phi_\lambda(\lambda(h)\, t) = \lambda(h)\, \Phi_\lambda(t) = \lambda(h)\, e_\lambda. \end{aligned}$$

The other assertions are proved in a similar way.

Let us prove now that $e_\lambda = t_\lambda$ for all $\lambda \in G$. Let $\lambda \in G$. Then it may be easily checked that $u = \sum_{(t)} \lambda(t_{(1)})\, t_{(2)} \bar{\lambda}(t_{(3)})$ is an integral such that $\varepsilon(u) = \varepsilon(t) = 1$. If follows that $u = t$, and applying Φ_λ to both sides yields that $t_\lambda = e_\lambda$.

(ii) This is clear from

$$\Phi_\mu(e_\lambda) = \Phi_\mu(\Phi_\lambda(t)) = \Phi_{\mu\lambda}(t) = e_{\mu\lambda}.$$

(iii) By 4.0.3 of [S, p. 81], we have that $\lambda(S(h))=\bar{\lambda}(h)$ for all $h\in H$. Then we have

$$S(e_\lambda)=S(\Phi_\lambda(t))=S(\textstyle\sum_{(t)}\bar{\lambda}(t_{(1)})\,t_{(2)})=\sum_{(t)}\bar{\lambda}(t_{(1)})\,S(t_{(2)})$$
$$=\textstyle\Sigma_{(t)}\lambda(S(t_{(1)}))\,S(t_{(2)})=\Psi_{\bar{\lambda}}(S(t))=\Psi_{\bar{\lambda}}(t)=t_{\bar{\lambda}}.$$

The other assertion is proved in a similar way.

2. Induction and Coinduction From Invariants

Let H be a finite dimensional Hopf algebra and A an H-module algebra. If A^H denotes the ring of invariants of A, recall that for all $\sigma\in G=\mathrm{Alg}(H,k)$, $(-)_\sigma$: $A\ \#\ H$-mod $\to A^H$-mod is a left exact functor taking $M\in A\ \#\ H$-mod to M_σ.

Consider $N\in A^H$-mod. Then $\mathrm{Ind}(N)=A\otimes_{A^H}N$ has an A-module and H-module structure given by $u(a\otimes n)=ua\otimes n$ where $u\in A$ or $u\in H$, $a\in A$, $n\in N$. It is easy to see that $\mathrm{Ind}(N)$ is an $A\ \#\ H$-module, and Ind: A^H-mod $\to A\ \#\ H$-mod defines a functor, called the *induced functor*. Now if $N\in A^H$-mod, we put

$$\mathrm{Coind}(N)=\mathrm{Hom}_{A^H}(A,N)$$

endowed with a structure of A-module by putting $(af)(b)=f(ba)$, for $a,b\in A$, $f\in\mathrm{Coind}(N)$, and with a structure of H-module by putting $(hf)(b)=f(\bar{S}(h)b)$ where $h\in H$, $b\in A$, $f\in\mathrm{Coind}(N)$ (recall that $\bar{S}$ denotes the composition inverse of the antipode, which exists since H is finite dimensional). Let us check that $\mathrm{Coind}(N)$ is an $A\ \#\ H$-module. Only condition (1.2) needs to be checked. If $h\in H$, $a,b\in A$, $f\in\mathrm{Coind}(N)$, then we have

$$[h(bf)](a)=(bf)(\bar{S}(h)a)=f((\bar{S}(h)a)b)=f(\textstyle\sum_{(h)}\bar{S}(h_{(2)})[a(h_{(1)}b)])$$

(we applied Lemma 1.3(i) at the last step). On the other hand

$$\textstyle\sum_{(h)}(h_{(1)}b)(h_{(2)}f)(a)=\sum_{(h)}(h_{(2)}f)(a(h_{(1)}b))=\sum_{(h)}f(\bar{S}(h_{(2)})[a(h_{(1)}b)]).$$

Now if $u\colon N\to N'$ is A^H-linear, then we define $\mathrm{Coind}(u)$: $\mathrm{Coind}(N)\to\mathrm{Coind}(N')$ by $\mathrm{Coind}(u)(f)=u\circ f$. Clearly Coind: A^H-mod $\to A\ \#\ H$-mod defines a functor, called the *coinduced functor*.

The main result of this section is the following:

2.1. Theorem. *Let H be a finite dimensional Hopf algebra and A an H-module algebra. Then the following assertions hold for any $\sigma\in G=\mathrm{Alg}(H,k)$, the set of grouplike elements of H^*:*

(a) *the functor* $T_{\bar{\sigma}} \circ \mathrm{Ind}$ *is a left adjoint of the functor* $(-)_\sigma$ *(recall that* $\bar{\sigma}$ *is the convolution inverse of* σ*);*

(b) *if* H *is semisimple then the functor* $T_{\bar{\sigma}} \circ \mathrm{Coind}$ *is a right adjoint of the functor* $(-)_\sigma$. *Moreover* $(-)_\sigma \circ T_{\bar{\sigma}} \circ \mathrm{Coind} \cong 1_{A^H\text{-mod}}$.

Proof. (a) Let $M \in A \# H$-mod and $N \in A^H$-mod. We define the functorial morphisms

$$\alpha : \mathrm{Hom}_{A \# H}(T_{\bar{\sigma}}(A \otimes_{A^H} N), M) \to \mathrm{Hom}_{A^H}(N, M_\sigma)$$

$$\beta : \mathrm{Hom}_{A^H}(N, M_\sigma) \to \mathrm{Hom}_{A \# H}(T_{\bar{\sigma}}(A \otimes_{A^H} N), M)$$

as follows. If $u \in \mathrm{Hom}_{A \# H}(T_{\bar{\sigma}}(A \otimes_{A^H} N), M)$ and $n \in N$, we put

$$\alpha(u)(n) = u(1 \otimes n).$$

We show that $\alpha(u)(n) \in M_\sigma$. Let $h \in H$. Then

$$\begin{aligned} h \cdot (\alpha(u)(n)) &= h \cdot u(1 \otimes n) = u(h \cdot (1 \otimes n)) = u(\textstyle\sum_{(h)} \sigma(h_{(2)})\, h_{(1)}(1 \otimes n)) \\ &= u(\textstyle\sum_{(h)} \sigma(h_{(2)})(h_{(1)} \cdot 1 \otimes n)) = u(\textstyle\sum_{(h)} \sigma(h_{(2)})(\varepsilon(h_{(1)})\, 1 \otimes n)) \\ &= u(\sigma(\textstyle\sum_{(h)} h_{(2)} \varepsilon(h_{(1)}))(1 \otimes n)) = u(\sigma(h)(1 \otimes n)) \\ &= \sigma(h)\, u(1 \otimes n) = \sigma(h)(\alpha(u)(n)). \end{aligned}$$

It is clear that $\alpha(u)$ is A^H-linear.

Now if $v \in \mathrm{Hom}_{A^H}(N, M_\sigma)$, we put $\beta(v)(a \otimes n) = av(n)$. Since it is easy to check that $\beta(v)$ is A-linear, let us check that it is H-linear. Let $h \in H$. We have

$$\begin{aligned} \beta(v)(h \cdot a \otimes n) &= \beta(v)(\textstyle\sum_{(h)} \sigma(h_{(2)})(h_{(1)} a \otimes n)) \\ &= \textstyle\sum_{(h)} \sigma(h_{(2)})(h_{(1)} a)\, v(n) = \textstyle\sum_{(h)} (h_{(1)} a)\, \sigma(h_{(2)})\, v(n) \\ &= \textstyle\sum_{(h)} (h_{(1)} a)(h_{(2)} v(n)) = h(av(n)) = h(\beta(v)(a \otimes n)). \end{aligned}$$

Now

$$\alpha(\beta(v))(n) = \beta(v)(1 \otimes n) = 1v(n) = v(n)$$

and

$$\beta(\alpha(u))(a \otimes n) = a\alpha(u)(n) = au(1 \otimes n) = u(a \otimes n),$$

hence α and β are inverse one to another.

(b) Since H is finite dimensional and semisimple, we let, for each $\sigma \in G$, t_σ denote the σ-integral, as in Lemma 1.6.

Let $M \in A \# H$-mod, $N \in A^H$-mod. We define the functorial morphisms

$$\gamma : \mathrm{Hom}_{A\#H}(M, T_{\bar{\sigma}}(\mathrm{Hom}_{A^H}(A, N))) \to \mathrm{Hom}_{A^H}(M_\sigma, N)$$
$$\delta : \mathrm{Hom}_{A^H}(M_\sigma, N) \to \mathrm{Hom}_{A\#H}(M, T_{\bar{\sigma}}(\mathrm{Hom}_{A^H}(A, N)))$$

as follows. If $u \in \mathrm{Hom}_{A\#B}(M, T_{\bar{\sigma}}(\mathrm{Hom}_{A^H}(A, N)))$, then we put $\gamma(u)(m_\sigma) = u(m_\sigma)(1)$ for $m_\sigma \in M_\sigma$. It is easy to see that $\gamma(u)$ is A^H-linear.

Now if $v \in \mathrm{Hom}_{A^H}(M_\sigma, N)$, we put $\delta(v)(m)(a) = v(t_\sigma(am))$ for $m \in M$, $a \in A$. It may easily be checked that $\delta(v)(m)$ is A^H-linear, since v is A^H-linear and the elements of A^H commute with elements of H, and that $\delta(v)$ is A-linear. Let us show now that $\delta(v)$ is H-linear. Indeed, if $h \in H$ then $\delta(v)(hm)(a) = v(t_\sigma(a(hm)))$, and

$$\begin{aligned}
[h \cdot (\delta(v)(m))](a) &= \textstyle\sum_{(h)} \sigma(h_{(2)})\, \delta(v)(m)(\bar{S}(h_{(1)})a) \\
&= \textstyle\sum_{(h)} \sigma(h_{(2)})\, v(t_\sigma((\bar{S}(h_{(1)})a)m)) \\
&= v(\textstyle\sum_{(h)} t_\sigma \sigma(h_{(2)})((\bar{S}(h_{(1)})a)m)) \\
&= v(\textstyle\sum_{(h)} t_\sigma h_{(2)}((\bar{S}(h_{(1)})a)m)) \\
&= v(t_\sigma(a(hm))),
\end{aligned}$$

by Lemma 1.3(iv).

Now

$$\gamma(\delta(v))(m_\sigma) = \delta(v)(m_\sigma)(1) = v(t_\sigma m_\sigma) = v(\sigma(t_\sigma)\, m_\sigma) = v(m_\sigma),$$

and

$$\begin{aligned}
\delta(\gamma(u))(m)(a) &= \gamma(u)(t_\sigma(am)) = u(t_\sigma(am))(1) = (t_\sigma \cdot u(am))(1) \\
&= \textstyle\sum_{(t_\sigma)} \sigma(t_{\sigma(2)})\, u(am)(\bar{S}(t_{\sigma(1)}) \cdot 1) = \sum_{(t_\sigma)} \sigma(t_{\sigma(2)})\, u(am)(\varepsilon(t_{\sigma(1)}) \cdot 1) \\
&= \sigma(t_\sigma)\, u(am)(1) = u(am)(1) = (au(m))(1) = u(m)(a),
\end{aligned}$$

therefore γ and δ are inverse one to another.

We now prove the last part of the statement. If $N \in A^H$-mod, then we have the canonical morphism of A^H-modules

$$\Phi : [T_{\bar{\sigma}}(\mathrm{Hom}_{A^H}(A, N))]_\sigma \to N$$

defined by $\Phi(v) = v(1_A)$.

If t_σ denotes the σ-integral as in Lemma 1.6 and if t is the idempotent in the integral of H, then recall that $t_\sigma = \sum_{(t)} \bar{\sigma}(t_{(2)})\, t_{(1)}$. If $v \in [T_{\bar{\sigma}}(\mathrm{Hom}_{A^H}(A, N))]_\sigma$, then

$$t_\sigma \cdot v = \sigma(t_\sigma) v = v,$$

thus if $a \in A$, we have

$$v(a) = (t_\sigma \cdot v)(a) = (\sum_{(t)} \bar{\sigma}(t_{(2)})\, t_{(1)} \cdot v)(a) = \sum_{(t)} \bar{\sigma}(t_{(3)})\, \sigma(t_{(2)})\, v(\bar{S}(t_{(1)})a)$$
$$= \sum_{(t)} \varepsilon(t_{(2)})\, v(\bar{S}(t_{(1)})a) = v(\bar{S}(t)a) = v(ta) = (ta)\, v(1),$$

so Φ is injective. Let $n \in N$, and put $v(a) = (ta)n$ for each $a \in A$. It is obvious that v is A^H-linear, and if $h \in H$, then

$$(h \cdot v)(a) = \sum_{(h)} \sigma(h_{(2)})\, v(\bar{S}(h_{(1)})a) = \sum_{(h)} \sigma(h_{(2)})(t\bar{S}(h_{(1)})a)n$$
$$= \sum_{(h)} \sigma(h_{(2)})(t\varepsilon(h_{(1)})a)n = \sigma(h)(ta)n = \sigma(h)\, v(a).$$

On the other hand $v(1) = n$, hence Φ is surjective too.

2.2. *Remarks.* (1) Assertion (a) from Theorem 2.1 was known in case $\sigma = \varepsilon$ from [D2, DT]. We note that in this case the result is much more general: k does not need to be a field, H is arbitrary.

(2) From assertion (a) we obtain that for all $\sigma \in G$, the functors $(-)_\sigma$ and $(-)_\varepsilon \circ T_\sigma$ are isomorphic, since they are both right adjoints of the functor $T_{\bar{\sigma}} \circ \mathrm{Ind}$. Thus, it would have been enough to prove assertion (b) only for $\sigma = \varepsilon$.

(3) In case (b) (i.e., if H is semisimple), then we also have that $(-)_\sigma \circ T_{\bar{\sigma}} \circ \mathrm{Ind} \cong \mathbb{1}_{A^H\text{-mod}}$, since by [CF2, Proposition 1.4], $\mathbb{1}_{A^H\text{-mod}} \cong (-)_\varepsilon \circ \mathrm{Ind}$ and it follows that $\mathbb{1}_{A^H\text{-mod}} \cong (-)_\varepsilon \circ T_\sigma \circ T_{\bar{\sigma}} \circ \mathrm{Ind} \cong (-)_\sigma \circ T_{\bar{\sigma}} \circ \mathrm{Ind}$.

We now establish some applications of Theorem 2.1.

2.3. COROLLARY. (i) *Suppose that H is finite dimensional, and that A/A^H is right H^*-Galois (see* [CFM] *for the definition; we will also say that $A^H \subset A$ is a Galois extension). Then* Ind *induces an equivalence from a quotient category of* A^H-mod *to* $A \# H$-mod.

(ii) *If H is finite dimensional and semisimple, then* A^H-mod $\cong$ $A \# H$-mod$/\mathcal{J}$, *where $\mathcal{J}$ is a localizing subcategory.*

Proof. (i) By Theorem 1.2 of [CFM, 2(b)], A is a projective finitely generated right A^H-module, hence A is flat as a right A^H-module. From Theorem 1.2 of [CFM, 5] it follows that $\mathrm{Ind}\, o(-)^H \cong \mathbb{1}_{A \# H\text{-mod}}$. Note that this isomorphism is exactly the functorial morphism associated to the adjoint pair $\mathrm{Ind}, (-)^H$. By a well-known result of Gabriel (see [G, Proposition 5, p. 374; Fa, Proposition 15.18]), and since $\mathrm{Ind}\colon A^H\text{-mod} \to A \# H\text{-mod}$ (which is an exact functor) has a right adjoint $(-)^H$ (by Theorem 2.1(a)), and $\mathrm{Ind}\, o(-)^H \cong \mathbb{1}_{A \# H\text{-mod}}$, it follows that $\mathrm{Ker(Ind)} = \{N \in A^H\text{-mod} : A \otimes_{A^H} N = 0\}$ is a localizing subcategory of A^H-mod, and $A^H\text{-mod}/\mathrm{Ker(Ind)} \cong A \# H\text{-mod}$.

(ii) By Theorem 2.1(b), $(-)^H = (-)_\varepsilon : A \# H\text{-mod} \to A^H\text{-mod}$, which is an exact functor, has a right adjoint Coind, and $(-)^H \circ \text{Coind} \cong 1_{A^H\text{-mod}}$. By the result of Gabriel referred to above, $A^H\text{-mod} \cong A \# H\text{-mod}/\mathscr{J}$, where

$$J = \text{Ker}(-)^H = \{M \in A \# H\text{-mod} : M^H = 0\}.$$

2.4. *Remarks.* (1) Assertion (i) of the above corollary may be derived at once from 2.14 of [DT].

(2) Theorem 2.2 of [CFM] states that if A/A^H is right H^*-Galois and A has an element of trace 1, then A^H and $A \# H$ are Morita equivalent. Assertion (i) of Corollary 2.3 shows what happens if we drop the condition "A has an element of trace 1." Moreover, by [KT] (see also [CFM, Lemma 2.1(4)]), A has an element of trace 1 if and only if A is faithfully flat over A^H (in the Galois case) so the above mentioned result follows at once from Corollary 2.3(i), and the equivalent conditions of Theorem 2.2 of [CFM] are equivalent to "Ind: $A^H\text{-mod} \to A \# H\text{-mod}$ is an equivalence of categories." We also remark that under the hypotheses of Theorem 2.2 of [CFM] we obtain for each $\sigma \in G$ an equivalence between $A \# H$-mod and A^H-mod, namely $(-)_\sigma$, with inverse $T_{\bar\sigma} \circ \text{Ind}$. In particular, if $M \in A \# H$-mod, $\sigma \in G$, $M_\sigma = 0$, then $M = 0$.

(3) Let H be semisimple finite dimensional and A/A^H right H^*-Galois. Then A has an element of trace 1, so A^H and $A \# H$ are Morita equivalent (see also [CF1, Theorem 4]). Corollary 2.3(ii) shows what happens if we drop the Galois condition. Again the result may be retrieved directly to Corollary 2.3(ii), since if the extension is Galois, then $M \cong A \otimes_{A^H} M^H$ for all $M \in A \# H$-mod, so $\text{Ker}(-)^H = 0$.

2.5. COROLLARY. *If H is finite dimensional and semisimple, and A/A^H is right H^*-Galois, then* Ind $\cong$ Coind.

Proof. By Theorem 2.1(b), $(-)^H \circ \text{Coind} \cong 1_{A^H\text{-mod}}$. But $\text{Ind} \circ (-)^H \cong 1_{A \# H\text{-mod}}$, since the extension is Galois, so the result follows.

We finish this section by listing some applications of the coinduced functor. The first result is a Hopf algebra version of Proposition 1.1 of [N].

2.6. PROPOSITION. *Let H be finite dimensional and semisimple. By Theorem 2.1(b), for each $M \in A \# H$-mod we have a functorial morphism of $A \# H$-modules*

$$\nu(M) : M \to T_{\bar\sigma}(\text{Coind}(M_\sigma)),$$

where $\nu(M)(m)(a) = t_\sigma(am)$, for $m \in M$, $a \in A$, and where t_σ is the σ-integral as in Lemma 1.6. Then $\text{Im}(\nu(M))$ is an essential submodule $T_{\bar\sigma}(\text{Coind}(M_\sigma))$.

Proof. Let $f \in T_{\bar{\sigma}}(\mathrm{Coind}(M_\sigma))$, $f \neq 0$. Then there exists $a \in A$ such that $f(a) \neq 0$ and $f(a) = x \in M_\sigma$. We have that

$$\begin{aligned} v(M)(x)(\alpha) &= t_\sigma(\alpha x) = \textstyle\sum_{(t_\sigma)} (t_{\sigma(1)}\sigma(t_{\sigma(2)})\alpha)x = (\Psi_{\bar{\sigma}}(t_\sigma)\alpha)f(a) \\ &= (t_{\sigma\bar{\sigma}}\alpha)f(a) = (t\alpha)f(a) = f((t\alpha)a) = (af)(t\alpha) = (af)(\bar{S}(t)\alpha) \\ &= (af)(\bar{S}(\textstyle\sum_{(t_\sigma)} t_{\sigma(1)}\sigma(t_{\sigma(2)}))\alpha) = \textstyle\sum_{(t_\sigma)} \sigma(t_{\sigma(2)})(af)(\bar{S}(t_{\sigma(1)})\alpha) \\ &= (t_\sigma \cdot (af))(\alpha), \end{aligned}$$

thus $t_\sigma \cdot (af) \in \mathrm{Im}(v(M))$. On the other hand $t_\sigma \cdot (af) \neq 0$ since

$$t_\sigma \cdot (af)(1) = (af)(t1_A) = (af)(1_A) = f(a) = x \neq 0.$$

Now we can use the Coind functor to describe injective envelopes in $A \# H$-mod, as in Section 1 of [N].

If H is finite dimensional and semisimple, and $\sigma \in G = \mathrm{Alg}(H, k)$, we put

$$\mathscr{C}_\sigma = \{M \in A \# H\text{-mod} : M_\sigma = 0\}.$$

Since $(-)_\sigma$ is an exact functor, it follows that $\mathscr{C}_\sigma$ is a localizing subcategory of $A \# H$-mod. We will say that $M \in A \# H$-mod is σ-faithful if $\tau_\sigma(M) = 0$, where τ_σ is the torsion radical associated to the localizing subcategory $\mathscr{C}_\sigma$. By Remark 2.4, if A/A^H is right H^*-Galois, then M is σ-faithful, for all $\sigma \in G (M \in A \# H\text{-mod})$.

The following is an immediate consequence of Theorem 2.1. The proof is the same as the proof of Corollary 1.1 of [N].

2.7. COROLLARY. *The following assertions hold if H is finite dimensional:*

(i) *if H is semisimple and N is an injective A^H-module, then* $\mathrm{Coind}(N)$ *is an injective $A \# H$-module;*

(ii) *if A is flat as a right A^H-module, and $M \in A \# H$*-mod *is injective, then M_σ is an injective A^H-module for each $\sigma \in G$.*

The proof of the result is a mere transcription of the proofs of Corollaries 1.2, 1.3, and 1.4 of [N], but we repeat it for the reader's convenience.

2.8. COROLLARY. *Let H be finite dimensional and semisimple. Then the following assertions hold*

(i) *if $Q \in A \# H$*-mod *is injective and σ-faithful, then Q_σ is injective over A^H, and*

$$Q \cong T_{\bar{\sigma}}(\mathrm{Coind}(Q_\sigma));$$

(ii) *if* $M \in A \# H$-mod *is* σ-*faithful, then*

$$E_{A \# H}(M) \cong T_{\bar{\sigma}}(\mathrm{Coind}(E_{A^H}(M_\sigma)))$$

(*here* $E_S(X)$ *denotes the injective envelope of the* S-*module* X);

(iii) *if* $\sum \in A \# H$-mod *is simple, then if* $\sum_\sigma \neq 0$, $\sum$ *is* σ-*faithful, and in this case*

$$E_{A \# H}(\textstyle\sum) \cong T_{\bar{\sigma}}(\mathrm{Coind}(E_{A^H}(\textstyle\sum_\sigma))).$$

Proof. (i) We show that the canonical morphism $\nu(Q): Q \to T_{\bar{\sigma}}(\mathrm{Coind}(Q_\sigma))$ is injective. Indeed, if $x_\sigma \in \mathrm{Ker}(\nu(Q))_\sigma$, then $\nu(Q)(x_\sigma) = 0$, hence

$$\nu(Q)(x_\sigma)(1) = t_\sigma x_\sigma = \sigma(t_\sigma)\, x_\sigma = x_\sigma = 0.$$

Thus $\mathrm{Ker}(\nu(Q))_\sigma = 0$, and $\mathrm{Ker}(\nu(Q)) \in \mathscr{C}_\sigma$. Consequently $\mathrm{Ker}(\nu(Q)) \subset \tau_\sigma(Q) = 0$, and $\nu(Q)$ is injective. By Proposition 2.6, $\nu(Q)$ is an isomorphism.

Let us show that Q_σ is injective over A^H. If $E(Q_\sigma) = E_{A^H}(Q_\sigma)$, then, since Coind is left exact, we have the monomorphism $\mathrm{Coind}(Q_\sigma) \to \mathrm{Coind}(E(Q_\sigma))$. But $\mathrm{Coind}(Q_\sigma) \cong T_\sigma(Q)$ as we just have shown, hence $\mathrm{Coind}(Q_\sigma)$ is an injective $A \# H$-module. Therefore, $\mathrm{Coind}(Q_\sigma)$ has an $A \# H$-complement in $\mathrm{Coind}(E(Q_\sigma))$, i.e.,

$$\mathrm{Coind}(E(Q_\sigma)) \cong \mathrm{Coind}(Q_\sigma) \oplus X$$

for some $X \in A \# H$-mod. Thus

$$E(Q_\sigma) \cong \mathrm{Coind}(E(Q_\sigma))^H \cong (\mathrm{Coind}(Q_\sigma))^H \oplus X^H \cong Q_\sigma \oplus X^H.$$

But $E(Q_\sigma)$ is an essential extension of Q_σ, so $X^H = 0$, hence $Q_\sigma \cong E(Q_\sigma)$ and Q_σ is injective over A^H.

(ii) Since the class of torsion free modules is closed under injective envelopes, it follows that $E_{A \# H}(M) \cong T_{\bar{\sigma}}(\mathrm{Coind}((E_{A \# H}(M))_\sigma))$, and $(E_{A \# H}(M))_\sigma$ is injective over A^H. Since $E_{A \# H}(M)$ is σ-faithful, M_σ is an essential A^H-submodule of $(E_{A \# H}(M))_\sigma$, hence

$$E_{A^H}(M_\sigma) = (E_{A \# H}(M))_\sigma.$$

(iii) Since $\sum_\sigma \neq 0$, then $\tau_\sigma(\sum) \neq 0$, otherwise $\sum = \tau_\sigma(\sum)$, and $\sum_\sigma = 0$, a contradiction.

We end this section by some remarks on Clifford Theory. First, we have, as in the graded case, the following:

2.9. COROLLARY. *Suppose H is finite dimensional and semisimple and let $\sum \in A \# H$-mod be a simple module. Let $\sigma \in G$ be such that $\sum_\sigma \neq 0$. Then the following assertions hold:*

(i) *$\sum_\sigma$ is a simple A^H-module;*

(ii) $\mathrm{End}_{A^H}(\sum_\sigma) \cong \mathrm{End}_{A \# H}(\sum)$.

Proof. (i) Let $x \in \sum_\sigma$, $x \neq 0$. Then Ax is a non-zero $A \# H$-submodule of $\sum$, so $Ax = \sum$. It is clear that $A^H x \subset \sum_\sigma$. Conversely, let $y \in \sum_\sigma$. Then $y = ax$ for some $a \in A$. If t_σ is the σ-integral from Lemma 1.6, then it may be easily seen that $y = t_\sigma y = t_\sigma(ax) = (ta)x \in A^H x$, so $A^H x = \sum_\sigma$.

(ii) The proof is identical to the proof of [GN, I3], and it goes as follows: since S is σ-faithful, we have an essential monomorphism

$$0 \to \textstyle\sum \to T_{\bar\sigma}(\mathrm{Coind}(\sum_\sigma)).$$

Let $\Phi: \mathrm{End}_{A \# H}(\sum) \to \mathrm{End}_{A^H}(\sum_\sigma)$ be the morphism given by $\Phi(f) = f^\sigma = f|_{\sum_\sigma}$. If $f, g \in \mathrm{End}_{A \# H}(\sum)$ such that $f^\sigma = g^\sigma$, let $x \in \sum_\sigma$, $x \neq 0$. Then $\sum = Ax$, so for all $y \in \sum$ there is an $a \in A$ such that $y = ax$. Hence $f(y) = af(x) = ag(x) = g(y)$, i.e., Φ is injective.

Now, for $0 \neq h \in \mathrm{End}_{A^H}(\sum_\sigma)$, we take $\bar h \in \mathrm{End}_{A \# H}(T_{\bar\sigma}(\mathrm{Coind}(\sum_\sigma)))$ to be the natural image of h. In order to show that Φ is surjective, it is enough to show that $\bar h(\sum) \subset \sum$ (in this case $\Phi(\bar h) = h$). We have the following situation: Y is an essential extension of X, X is simple, and u is an automorphism of Y. We want that $u(X) \subset X$, but that is clear, and the proof is complete.

Let now $\sum \in A \# H$-mod be a simple module, and denote by

$$\mathrm{Mod}(A \mid \textstyle\sum) = \{X \in A\text{-mod} \mid \sum^{(I)} \to X \to 0\}$$

the full subcategory of A-mod consisting of all A-modules which are $\sum$-generated. The Direct Clifford Theory (see [Da]) is an equivalence of categories between $\mathrm{Mod}(A \mid \sum)$ and $\mathrm{End}_A(\sum)$-mod. Trying to prove this in our case turns out to be difficult from the very beginning, since (unlike the graded case) it is not clear at all whether $\sum$ is semisimple over A^H. Adding an extra condition may make things trivial. For example, if we require the action of H on A to be inner, then $A \subset A \# H$ is a centralizing extension, so $\sum$ is semisimple over A, like all objects of $\mathrm{Mod}(A \mid \sum)$. Then it is clear that $\sum$ is a small projective generator in $\mathrm{Mod}(A \mid \sum)$.

3. INDUCTION AND COINDUCTION FROM SEMI-INVARIANTS

Let H be a finite dimensional Hopf algebra and A an H-module algebra. Throughout this section, G will denote the group of grouplike elements of

H^*, $G = \mathrm{Alg}(H, k)$, $S_A = \bigoplus_{\lambda \in G} A_\lambda$ the ring of semi-invariants of A which is a graded ring of type G, S_A-gr the category of left G-graded S_A-modules. We will denote by $S_{(-)}$: $A \# H\text{-mod} \to S_A\text{-gr}$ the functor taking $M \in A \# H$-mod to the module of semi-invariants $S_M = \bigoplus_{\lambda \in G} M_\lambda$, which is a left exact functor.

If $N \in S_A$-gr, $N = \bigoplus_{\lambda \in G} N_\lambda$ then we define the $A \# H$-module *induced* by N as follows: $\mathrm{IND}(N) = A \otimes_{S_A} N$, which becomes an $A \# H$-module if we put

$$b(a \otimes n) = ba \otimes n \qquad \text{and} \qquad h(a \otimes n_\lambda) = \textstyle\sum_{(h)} \lambda(h_{(2)})\, h_{(1)}a \otimes n_\lambda,$$

where $a, b \in A$, $n \in N$, $h \in H$, and $n_\lambda \in N_\lambda$. We have

$$\begin{aligned}\textstyle\sum_{(h)} (h_{(1)}a)(h_{(2)}(b \otimes n_\lambda)) &= \textstyle\sum_{(h)} (h_{(1)}a)\, \lambda(h_{(3)})\, h_{(2)}b \otimes n_\lambda \\ &= \textstyle\sum_{(h)} \lambda(h_{(2)})\, h_{(1)}(ab) \otimes n_\lambda \\ &= h(ab \otimes n_\lambda) = h(a(b \otimes n_\lambda))\end{aligned}$$

and hence $\mathrm{IND}(N)$ is an $A \# H$-module. Clearly $\mathrm{IND}: S_A\text{-gr} \to A \# H\text{-mod}$ defines a functor.

The *coinduced* $A \# H$-module by $N \in S_A$-gr is defined as

$$\mathrm{COIND}(N) = \mathrm{Hom}_{S_A}(A, N)$$

which becomes an $A \# H$-module if we put for $a, b \in A$, $h \in H$ and $f \in \mathrm{Hom}_{S_A}(A, N)$,

$$(af)(b) = f(ba); \qquad (hf)(b) = \textstyle\sum_{\lambda \in G} [f(\sum_{(h)} \bar{S}(h_{(1)})\, \lambda(h_{(2)})b)]_\lambda$$

(recall that if $n \in N$, then $n_\lambda \in N_\lambda$ is the homogeneous component of n of degree λ). If $g \in H$, then

$$\begin{aligned}[(h(gf))(b)]_\lambda &= [f(\textstyle\sum_{(h),(g)} \bar{S}(g_{(1)})\, \bar{S}(h_{(1)})\, \lambda(h_{(2)})\, \lambda(g_{(2)})b)]_\lambda \\ &= [f(\textstyle\sum_{(h),(g)} \bar{S}(h_{(1)}g_{(1)})\, \lambda(h_{(2)}g_{(2)})b)]_\lambda \\ &= [((hg)f)(b)]_\lambda\end{aligned}$$

for all $\lambda \in G$.

Now if $h \in H$, $a, b \in A$, $f \in \mathrm{COIND}(N)$, then we have

$$\begin{aligned}(h(bf))(a) &= \textstyle\sum_{\lambda \in G} [f((\sum_{(h)} \bar{S}(h_{(1)})\, \lambda(h_{(2)})a)b)]_\lambda \\ &= \textstyle\sum_{\lambda \in G} [f(\sum_{(h)} \lambda(h_{(3)})\, \bar{S}(h_{(2)})(a(h_{(1)}b)))]_\lambda\end{aligned}$$

(by Lemma 1.3(i)). On the other hand,

$$\sum_{(h)} ((h_{(1)}b)(h_{(2)}f))(a) = \sum_{(h)} (h_{(2)}f)(a(h_{(1)}b))$$
$$= \sum_{\lambda \in G} [f(\sum_{(h)} \bar{S}(h_{(2)})\, \lambda(h_{(3)})(a(h_{(1)}b)))]_{\lambda},$$

so COIND(N) is an $A \# H$-module. Clearly COIND: $S_A \to A \# H$-mod defines a functor.

The main result of this section is the following analogue of Theorem 2.1.

THEOREM. *Let H be a finite dimensional Hopf algebra and A an H-module algebra. Then the following assertions hold:*

(1) IND *is a left adjoint of the functor* $S_{(-)}$;

(2) *if H is semisimple, then* COIND *is a right adjoint of* $S_{(-)}$. *Moreover,*

$$S_{(-)} \circ \mathrm{COIND} \cong 1_{S_A\text{-gr}},$$

and

$$S_{(-)} \circ \mathrm{IND} \cong 1_{S_A\text{-gr}}.$$

Proof. (1) If $M \in A \# H$-mod and $N \in S_A$-gr, $N = \bigoplus_{\lambda \in G} N_\lambda$, we define the functorial morphisms

$$\alpha : \mathrm{Hom}_{A \# H}(A \otimes_{S_A} N, M) \to \mathrm{Hom}_{S_A\text{-gr}}(N, S_M)$$
$$\beta : \mathrm{Hom}_{S_A\text{-gr}}(N, S_M) \to \mathrm{Hom}_{A \# H}(A \otimes_{S_A} N, M)$$

as follows. If $u \in \mathrm{Hom}_{A \# H}(A \otimes_{S_A} N, M)$ and $n \in N_\lambda$, we put $\alpha(u)(n_\lambda) = u(1 \otimes n_\lambda)$. Let us check that $\alpha(u)$ is a morphism in S_A-gr. First, $u(1 \otimes n_\lambda) \in M_\lambda$ because if $h \in H$, then

$$hu(1 \otimes n_\lambda) = u(h(1 \otimes n_\lambda)) = u(\sum_{(h)} \lambda(h_{(2)})\, h_{(1)} 1 \otimes n_\lambda)$$
$$= u(\sum_{(h)} \lambda(h_{(2)})\, \varepsilon(h_{(1)})\, 1 \otimes n_\lambda) = u(\lambda(\sum_{(h)} h_{(2)} \varepsilon(h_{(1)}))\, 1 \otimes n_\lambda)$$
$$= u(\lambda(h)\, 1 \otimes n_\lambda) = \lambda(h)\, u(1 \otimes n_\lambda).$$

It is clear that $\alpha(u)$ is S_A-linear.

Now for $v \in \mathrm{Hom}_{S_A\text{-gr}}(N, S_M)$, we put $\beta(v)(a \otimes n_\lambda) = av(n_\lambda)$, for $a \in A$, $n_\lambda \in N_\lambda$. It is clear that $\beta(v)$ is A-linear. Let us show that it is H-linear too. Take $h \in H$. We have that

$$\beta(v)(h(a \otimes n_\lambda)) = \beta(v)(\sum_{(h)} \lambda(h_{(2)})\, h_{(1)} a \otimes n_\lambda)$$
$$= \sum_{(h)} \lambda(h_{(2)})(h_{(1)}a)\, v(n_\lambda) = \sum_{(h)} (h_{(1)}a)(h_{(2)}\, v(n_\lambda))$$
$$= h(av(n_\lambda)) = h(\beta(v)(a \otimes n_\lambda)).$$

Now

$$\alpha(\beta(v))(n_\lambda)=\beta(v)(1\otimes n_\lambda)=1\cdot v(n_\lambda)=v(n_\lambda)$$

and

$$\beta(\alpha(u))(a\otimes n_\lambda)=a\alpha(u)(n_\lambda)=au(1\otimes n_\lambda)=u(a\otimes n_\lambda),$$

hence α and β are inverse to one another.

(2) Since H is semisimple, let for each $\lambda\in G$, e_λ (or t_λ) denote the λ-integrals as in Lemma 1.6. Recall that $e_\varepsilon=t_\varepsilon=t$, the idempotent in the integral of H. Let $M\in A\ \#\ H$-mod, $N\in S_A$-gr, $N=\bigoplus_{\lambda\in G}N_\lambda$ and define the functorial morphisms

$$\gamma:\mathrm{Hom}_{A\,\#\,H}(M,\mathrm{Hom}_{S_A}(A,N))\to\mathrm{Hom}_{S_A\text{-gr}}(S_M,N)$$
$$\delta:\mathrm{Hom}_{S_A\text{-gr}}(S_M,N)\to\mathrm{Hom}_{A\,\#\,H}(M,\mathrm{Hom}_{S_A}(A,N))$$

as follows. If $u\in\mathrm{Hom}_{A\,\#\,H}(M,\mathrm{Hom}_{S_A}(A,N))$, then

$$\gamma(u)(m_\lambda)=[u(m_\lambda)(1)]_\lambda$$

for each $m_\lambda\in M_\lambda$. Let us show that $\gamma(u)$ is a morphism in S_A-gr. Let $a_\sigma\in A_\sigma$. We have

$$\begin{aligned}\gamma(u)(a_\sigma m_\lambda)&=[u(a_\sigma m_\lambda)(1)]_{\sigma\lambda}=[(a_\sigma u(m_\lambda)(1))]_{\sigma\lambda}=[u(m_\lambda)(a_\sigma)]_{\sigma\lambda}\\&=[a_\sigma(u(m_\lambda)(1))]_{\sigma\lambda}=a_\sigma[u(m_\lambda)(1)]_\lambda=a_\sigma\gamma(u)(m_\lambda).\end{aligned}$$

Now if $v\in\mathrm{Hom}_{S_A\text{-gr}}(S_M,N)$, we put $[\delta(v)(m)(a)]_\lambda=v(e_\lambda(am))$. It is easy to see that $\delta(v)$ is A-linear. Let us show that it is H-linear. Let $h\in H$. We have

$$[\delta(v)(hm)(a)]_\lambda=v(e_\lambda(a(hm)))$$

and

$$\begin{aligned}[(h\delta(v)(m)(a)]_\lambda&=[\delta(v)(m)(\textstyle\sum_{(h)}\bar{S}(h_{(1)})\,\lambda(h_{(2)})a)]_\lambda\\&=v(e_\lambda((\textstyle\sum_{(h)}\bar{S}(h_{(1)})\,\lambda(h_{(2)})a)m)).\end{aligned}$$

But

$$e_\lambda(a(hm))=e_\lambda(\textstyle\sum_{(h)}h_{(2)}[\bar{S}(h_{(1)})a)m])=\sum_{(h)}e_\lambda\lambda(h_{(2)})(\bar{S}(h_{(1)})a)m$$

(we used Lemma 1.6(ii)), therefore

$$\begin{aligned}\delta(v)(m)(a_\sigma a)&=\textstyle\sum_{\lambda\in G}[\delta(v)(m)(a_\sigma a)]_\lambda=\sum_{\lambda\in G}a_\sigma[\delta(v)(m)(a)]_{\bar{\sigma}\lambda}\\&=a_\sigma\textstyle\sum_{\lambda\in G}[\delta(v)(m)(a)]_{\bar{\sigma}\lambda}=a_\sigma\,\delta(v)(m)(a).\end{aligned}$$

Now

$$\gamma(\delta(v))(m_\lambda)=[\delta(v)(m_\lambda)(1)]_\lambda=v(e_\lambda m_\lambda)=v(\lambda(e_\lambda)\,m_\lambda)=v(1m_\lambda)=v(m_\lambda),$$

and, on the other hand,

$$\begin{aligned}[\delta(\gamma(u))(m)(a)]_\lambda&=\gamma(u)(e_\lambda(am))=[u(e_\lambda(am))(1)]_\lambda=[(e_\lambda u(am))(1)]_\lambda\\&=[u(am)(\textstyle\sum_{(e_\lambda)}\bar{S}(e_{\lambda_{(1)}})\,\lambda(e_{\lambda_{(2)}})\cdot 1)]_\lambda\\&=[u(am)(\textstyle\sum_{(e_\lambda)}\lambda(e_{\lambda_{(2)}})\,\varepsilon(\bar{S}(e_{(1)}))\cdot 1)]_\lambda\\&=[u(am)(\lambda(\textstyle\sum_{(e_\lambda)}e_{\lambda_{(2)}}\varepsilon(e_{\lambda_{(1)}})\cdot 1)]_\lambda\\&=[u(am)(\lambda(e_\lambda)\cdot 1)]_\lambda=[u(am)(1)]_\lambda\\&=[au(m)(1)]_\lambda=[u(m)(a)]_\lambda,\end{aligned}$$

hence γ and δ are inverse to one another.

If $N\in S_A$-gr, we have the canonical morphism in S_A-gr

$$\Phi\colon S_{\mathrm{Hom}_{S_A}(A,N)}\to N$$

given by $\Phi(v_\lambda)=[v_\lambda(1)]_\lambda$ for $v_\lambda\in\mathrm{Hom}_{S_A}(A,N)_\lambda$. Since $v_\lambda\in\mathrm{Hom}_{S_A}(A,N)_\lambda$, if $t_\lambda=\sum_{(t)}t_{(1)}\bar{\lambda}(t_{(2)})$ is the λ-integral of Lemma 1.6, then $t_\lambda v_\lambda=\lambda(t_\lambda)\,v_\lambda=v_\lambda$.

Thus, for $a\in A$ we have

$$\begin{aligned}v_\lambda(a)=(t_\lambda v_\lambda)(a)&=\textstyle\sum_{\mu\in G}[v_\lambda(\sum_{(t)}\bar{S}(t_{(1)})\,\mu(t_{(2)})\,\bar{\lambda}(t_{(3)})a)]_\mu\\&=\textstyle\sum_{\mu\in G}[v_\lambda(\sum_{(t)}\bar{S}(t_{(1)})\,\mu\bar{\lambda}(t_{(2)})a)]_\mu\\&=\textstyle\sum_{\mu\in G}[v_\lambda(\bar{S}(t_{\lambda\bar{\mu}})a)]_\mu\\&=\textstyle\sum_{\mu\in G}[v_\lambda(e_{\mu\bar{\lambda}}a)]_\mu\qquad\text{(by Lemma 1.6(iii))}\\&=\textstyle\sum_{\mu\in G}[v_\lambda(e_{\mu\bar{\lambda}})a)\,v_\lambda(1)]_\mu\\&=\textstyle\sum_{\mu\in G}(e_{\mu\bar{\lambda}}a)[v_\lambda(1)]_{\lambda\bar{\mu}\mu}\\&=\textstyle\sum_{\mu\in G}(e_\mu a)[v_\lambda(1)]_\lambda,\end{aligned}$$

hence Φ is injective. Now if $n\in N_\lambda$, we put $v_\lambda(a)=\sum_{\mu\in G}(e_\mu a)n$. If $a_\sigma\in A_\sigma$, we have

$$\begin{aligned}v_\lambda(a_\sigma a)&=\textstyle\sum_{\mu\in G}(e_\mu(a_\sigma a))n\\&=\textstyle\sum_{\mu\in G}[\sum_{(e_\mu)}\sigma(e_{\mu_{(1)}})\,a_\sigma(e_{\mu_{(2)}}a)]n\\&=\textstyle\sum_{\mu\in G}[a_\sigma(\Phi_{\bar{\sigma}}(e_\mu)a]n\\&=\textstyle\sum_{\mu\in G}[a_\sigma(e_{\bar{\sigma}\mu}a)]n\\&=a_\sigma(\textstyle\sum_{\mu\in G}(e_\mu a)n)\\&=a_\sigma v_\lambda(a),\end{aligned}$$

then $v_\lambda \in \mathrm{Hom}_{S_A}(A, N)$. If $h \in H$, then

$$\begin{aligned}[(hv_\lambda)(a)]_\mu &= [v_\lambda(\textstyle\sum_{(h)} \bar{S}(h_{(1)})\, \mu(h_{(2)})a)]_\mu = [e_{\mu\bar{\lambda}} \textstyle\sum_{(h)} \bar{S}(h_{(1)})\, \mu(h_{(2)})a]\,n \\ &= [\textstyle\sum_{(h)} e_{\mu\bar{\lambda}} \mu\bar{\lambda}(\bar{S}(h_{(1)}))\, \mu(h_{(2)})a]\,n \\ &= [\textstyle\sum_{(h)} e_{\mu\bar{\lambda}} \lambda\bar{\mu}(h_{(1)})\, \mu(h_{(2)})a]\,n \\ &= (e_{\mu\bar{\lambda}} \lambda(h)a)n = \lambda(h)(e_{\mu\bar{\lambda}}a)n = \lambda(h)[v_\lambda(a)]_\mu,\end{aligned}$$

so $v_\lambda \in \mathrm{Hom}_{S_A}(A, N)_\lambda$. Moreover $[v_\lambda(1)]_\lambda = (e_{\lambda\bar{\lambda}}1)n = (e_\varepsilon 1)n = (t \cdot 1)n = (\varepsilon(t)1)n = 1 \cdot n = n$, so Φ is surjective too.

Finally, let us prove the last statement. For al $N \in S_A$-gr, we have the canonical graded morphism of S_A-modules

$$\Psi: N \to S_{A \otimes_{S_A} N}$$

given by

$$\Psi(n_\lambda) = 1 \otimes n_\lambda$$

for all $n_\lambda \in N_\lambda$. Now if $a \otimes n \in (A \otimes_{S_A} N)_\sigma$, then

$$\begin{aligned}a \otimes n = t_\sigma(a \otimes n) &= t_\sigma(\textstyle\sum_{\lambda \in G} a \otimes n_\lambda) = \textstyle\sum_{\lambda \in G} t_\sigma(a \otimes n_\lambda) \\ &= \textstyle\sum_{\lambda \in G} \Psi_\lambda(t_\sigma) a \otimes n_\lambda = \textstyle\sum_{\lambda \in G} t_{\sigma\bar{\lambda}} a \otimes n_\lambda \\ &= \textstyle\sum_{\lambda \in G} 1 \otimes (t_{\sigma\bar{\lambda}} a)\, n_\lambda = 1 \otimes \textstyle\sum_{\lambda \in G} (t_{\sigma\bar{\lambda}} a)\, n_\lambda.\end{aligned}$$

Thus we define $\Psi': S_{A \otimes_{S_A} N} \to N$ by

$$\Psi'(a \otimes n) = \textstyle\sum_{\lambda \in G} (t_{\sigma\bar{\lambda}} a)\, n_\lambda \in N_\sigma$$

for $a \otimes n \in (A \otimes_{S_A} N)_\sigma$. It is now easy to check that Ψ and Ψ' are mutually inverse. Being the inverse of a bijective S_A-linear map, Ψ' is S_A-linear.

3.2. COROLLARY. *If H is finite dimensional and semisimple, and A is an H-module algebra, then $S_{(-)}$ induces an equivalence of categories between A # H-mod/Ker($S_{(-)}$) and S_A # kG^*-mod.*

Proof. Since $S_{(-)}$ is exact, A # H-mod/Ker($S_{(-)}$) is equivalent to S_A-gr by the result of Gabriel referred to in the proof of Corollary 2.3(i). On the other hand, S_A-gr is isomorphic to S_A # kG^*-mod by Theorem 2.2 of [CM] or by Proposition 1.2.

Remarks. As the referee kindly pointed out, if we add to the hypothesis of Corollary 3.2 "A/A^H is H^*-Galois," then we obtain that S_A is a strongly graded ring. This is a particular case of Remark 3.11(2) of [Sc] and also a generalization of Ulbrich's result [U] which states that a ring graded by a finite group is strongly graded if and only if it is a Galois extension (in the Hopf algebra sense) of the part of degree 1. Perhaps Proposition 2.4 of [VO] might be used in conjunction with this to prove results on finiteness conditions for Galois extensions in the semisimple case. We also remark that in this case IND ≅ COIND.

ACKNOWLEDGMENTS

The second author thanks his co-authors' universities for their hospitality while this research was done.

The authors thank Susan Montgomery and the referee for their interesting remarks.

REFERENCES

[BCM] R. BLATTNER, M. COHEN, AND S. MONTGOMERY, Crossed products and linear actions of Hopf algebras, *Trans. Amer. Math. Soc.* **298** (1986), 671–711.

[BM] R. BLATTNER AND S. MONTGOMERY, Crossed products and Galois extensions of Hopf algebras, *Pacific J. Math.* **137** (1989), 37–54.

[C] M. COHEN, Smash products, inner actions and quotient rings, *Pacific J. Math.* **125** (1986), 45–66.

[CF1] M. COHEN AND D. FISHMAN, Hopf algebra actions, *J. Algebra* **100** (1986), 363–379.

[CF2] M. COHEN AND D. FISHMAN, Semisimple extensions and elements of trace 1, to appear.

[CFM] M. COHEN, D. FISHMAN, AND S. MONTGOMERY, Hopf Galois extensions, smash products and Morita equivalence, *J. Algebra* **113** (1990), 351–372.

[CM] M. COHEN AND S. MONTGOMERY, Group graded rings, Smash products and group actions, *Trans. Amer. Math. Soc.* **282** (1984), 237–258.

[Da] E. C. DADE, Clifford theory for group-graded rings, II, *J. Reine Angew. Math.* **387** (1988), 148–181.

[D1] Y. DOI, On the structure of relative Hopf modules, *Comm. Algebra* 11 (1983), 243–255.

[D2] Y. DOI, Algebras with total integrals, *Comm. Algebra* **12** (1984), 1155–1169.

[D3] Y. DOI, Hopf extensions of algebras and Maschke-type theorems, *Israel J. Math.* **72** (1990), 99–108.

[DT] Y. DOI AND M. TAKEUCHI, Hopf–Galois extensions of algebras, the Miyashita–Ulbrich action and Azumaya algebras, *J. Algebra* **121** (1989), 488–516.

[Fa] C. FAITH, "Algebra: Rings, Modules and Categories, I and II," Springer-Verlag, Berlin, 1973.

[Fi] D. FISHMAN, Ph.D. Thesis, 1989.

[G] P. GABRIEL, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[GN] J. L. GOMEZ PARDO AND C. NĂSTĂSESCU, Relative projectivity, Clifford theory and applications, *J. Algebra* **141** (1991), 484–504.

[KT] H. F. KREIMER AND M. TAKEUCHI, Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* **30** (1981), 675–692.

[N] C. NĂSTĂSESCU, Some constructions over graded rings: Applications, *J. Algebra* **120** (1989), 119–138.

[NV] C. NĂSTĂSESCU AND F. VAN OYSTAEYEN, "Graded Ring Theory," North-Holland, Amsterdam, 1982.

[S] M. SWEEDLER, "Hopf Algebras," Benjamin, New York, 1969.

[Sc] H. J. SCHNEIDER, Principal homogeneous spaces for arbitrary Hopf algebras, *Israel J. Math.* **72** (1990), 167–195.

[U] K. H. ULBRICH, Vollgraduierte Algebren, *Abh. Math. Sem. Univ. Hamburg* **51** (1981), 136–148.

[VO] F. VAN OYSTAEYEN, Derivations of graded rings and Clifford systems, *J. Algebra* **98** (1986), 485–498.