

# DUALITY FOR FINITE HOPF ALGEBRAS EXPLAINED BY CORINGS

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**ABSTRACT.** We give a coring version for the duality theorem for actions and coactions of a finitely generated projective Hopf algebra. We also provide a coring analogue for a theorem of H.-J. Schneider, which generalizes and unifies the duality theorem for finite Hopf algebras and its refinements.

## INTRODUCTION

The Duality Theorem for actions and coactions of a finite Hopf algebra is the following:

**Theorem 0.1.** [2, 12] *Let  $H$  be a finitely generated projective Hopf algebra over the commutative ring  $k$ , and  $A$  a right  $H$ -comodule algebra. Then*

$$(A \# H^*) \# H \simeq \text{End}_A(H \otimes A).$$

The following are two refinements of Theorem 0.1:

**Theorem 0.2.** [11] *Let  $H$  be a finitely generated projective Hopf algebra over the commutative ring  $k$ , and  $A$  a right  $H$ -comodule algebra. Then:*

- i)  $\text{End}_A^H(H \otimes A) \# H \simeq \text{End}_A(H \otimes A)$
- ii)  $\text{End}_A^H(H \otimes A) \simeq A \# H^*$ .

It was shown in [5] that both assertions of the previous theorem are particular cases of the following result, due to H.-J. Schneider:

**Theorem 0.3.** [8] *Let  $H$  be a finitely generated projective Hopf algebra over the commutative ring  $k$ ,  $A$  a left  $H$ -module algebra, and  $B = A^H$  the algebra of invariants. Assume that  $A/B$  is a right  $H^*$ -Hopf Galois extension, and let  $M \in \mathcal{M}_A^{H^*}$ . Then  $\text{End}_A(M)$  is a left  $H$ -module algebra via the action  $(h \cdot f)(m) = h_{(1)} f(S(h_{(2)})m)$ , and we have the algebra isomorphism*

$$\text{End}_A(M) \# H \simeq \text{End}_B(M),$$

*sending  $f \# h$  to the endomorphism of  $M$  mapping  $m$  to  $f(h)m$ .*

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Finding the point of view that makes the proof of the finite duality theorem trivial has at least two possible answers: one could be “Hopf-Galois extensions”, as in [12], and the second one could be “the fundamental theorem for Hopf modules”, because this result was used in [5] to prove a more general form of Schneider’s theorem.

There is yet another question about the finite duality theorem that seems worth answering. There are apparently two distinct types of duality results.

The first one (like in Theorem 0.1) could be described as follows: take an algebra; let  $H^*$  act on it and take the smash product;  $H$  acts on it, so take the smash product again; get back to the algebra (actually to its endomorphism ring, which is Morita equivalent to it).

The second one is: take object; take dual; take dual again; get back to object.

By comparing the two, we see that the way to make them agree is to describe the smash product as the dual of an object. This object is a coring.

**Definition 0.4.** *Let  $A$  be a ring. An  $A$ -bimodule  $\mathcal{C}$  is called a coring if there exist  $A$ -bimodule maps  $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$  and  $\varepsilon : \mathcal{C} \longrightarrow A$ , such that  $\Delta$  is coassociative and  $\varepsilon$  is a counit.*

Coring were introduced by Sweedler in [10], and were given a lot of attention lately, after Takeuchi remarked that many examples of (generalized) Hopf modules are in fact just comodules over some corings. For all unexplained facts about corings the reader is referred to [3].

The most basic example of a coring is a coalgebra over a commutative ring  $A$ . Another fundamental example is the canonical coring associated to the ring homomorphism  $i : B \longrightarrow A$ :  $\mathcal{C} = A \otimes_B A$ ,  $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \simeq A \otimes_B A \otimes_B A$ ,  $\Delta(a \otimes b) = a \otimes 1 \otimes b$ , and  $\varepsilon : \mathcal{C} \longrightarrow A$ ,  $\varepsilon(a \otimes b) = ab$ .

**Example 0.5.** *Let  $H$  be a  $k$ -Hopf algebra, and  $A$  a right  $H$ -comodule algebra via  $a \mapsto a_{[0]} \otimes a_{[1]}$ . Then  $A \otimes H$  becomes an  $A$ -coring as follows: the left  $A$ -module structure is given by multiplication on the first component, and the right  $A$  module structure is given by  $(a \otimes h)b = ab_{[0]} \otimes hb_{[1]}$ . The comultiplication is  $\Delta : A \otimes H \longrightarrow (A \otimes H) \otimes_A (A \otimes H) \simeq A \otimes H \otimes H$ ,  $\Delta(a \otimes h) = a \otimes h_{(1)} \otimes h_{(2)}$ , and the counit is  $\varepsilon : A \otimes H \longrightarrow A$ ,  $\varepsilon(a \otimes h) = \varepsilon(h)a$ .*

Recall that if  $A$  is a right  $H$ -comodule algebra, then the smash product  $A \# H^*$  has multiplication defined by  $(a \# h^*)(b \# g^*) = ab_{[0]} \# (h^* \leftharpoonup b_{[1]})g^*$ . The big smash product  $\#(H, A) = \text{Hom}_k(H, A)$  has multiplication defined by  $(f \cdot g)(h) = f(g(h_{(2)})h_{(1)})g(h_{(2)})_0$ . If  $H$  is projective finitely generated over  $k$ , and  $h_i$  and  $h_i^*$ ,  $i = 1 \dots n$ , are dual bases, then  $\#(H, A) \simeq A \# H^*$  as rings, via  $f \mapsto \sum_{i=1}^n f(h_i) \# h_i^*$ . Note that if  $B = A^{coH}$ , then  $A/B$  is a right  $H^*$ -Hopf-Galois extension if and only if the coring  $A \otimes H$  from example 0.5 is isomorphic to the canonical coring  $A \otimes_B A$  via the map sending  $a \otimes b \in A \otimes_B A$  to  $ab_{[0]} \otimes b_{[1]} \in A \otimes H$ .

### 1. DUALITY FOR CORINGS

In this section we prove a duality theorem for finitely generated projective corings, and show that it extends the duality theorem for finite Hopf algebras.

Let  $\mathcal{C}$  be an  $A$ -coring. Then the left dual of  $\mathcal{C}$ , denoted by  ${}^*\mathcal{C}$ , is by definition

$${}^*\mathcal{C} = {}_A\text{Hom}(\mathcal{C}, A),$$

with multiplication given by

$$(\phi * \psi)(c) = \psi(c_{(1)}\phi(c_{(2)})).$$

Then  $A$  becomes a subring of  ${}^*\mathcal{C}$  via

$$i : A \longrightarrow {}^*\mathcal{C}, \quad i(a)(c) = \varepsilon(c)a.$$

Similarly, the right dual of  $\mathcal{C}$ , denoted by  $\mathcal{C}^*$ , is by definition

$$\mathcal{C}^* = \text{Hom}_A(\mathcal{C}, A),$$

with multiplication given by

$$(\phi * \psi)(c) = \phi(\psi(c_{(1)})c_{(2)}).$$

Now  $A$  becomes a subring of  $\mathcal{C}^*$  via

$$j : A \longrightarrow \mathcal{C}^*, \quad j(a)(c) = a\varepsilon(c).$$

**Theorem 1.1.** *Let  $\mathcal{C}$  be an  $A$ -coring, and assume that  $\mathcal{C}$  is finitely generated projective as a left  $A$ -module. Then the right dual of the canonical coring associated to the ring morphism  $i : A \longrightarrow {}^*\mathcal{C}$  is isomorphic to the (opposite) endomorphism ring  $\text{End}({}_A\mathcal{C})^{op}$ .*

*Proof.* We have the isomorphism

$$\alpha : \text{Hom}_{{}^*\mathcal{C}}({}^*\mathcal{C} \otimes_A {}^*\mathcal{C}, {}^*\mathcal{C}) \xrightarrow{\sim} \text{Hom}_A({}^*\mathcal{C}, {}^*\mathcal{C}) : \beta,$$

defined by  $\alpha(\phi)(c^*) = \phi(c^* \otimes 1)$  and  $\beta(\psi)(c^* \otimes d^*) = \psi(c^*)d^*$ . (Note that  $\alpha$  is simply the composition of the hom-tensor adjointness with the equivalence of  $\text{Hom}_R(R, -)$ , for  $R = {}^*\mathcal{C}$  an associative ring.) Now duality for finitely generated projective modules provides the isomorphism

$$\gamma : \text{Hom}_A({}^*\mathcal{C}, {}^*\mathcal{C}) \xrightarrow{\sim} {}_A\text{Hom}(\mathcal{C}, \mathcal{C}) : \delta,$$

which is defined as follows: let  $c = \sum_{i=1}^n f_i(c)c_i$  be the dual basis formula in  $\mathcal{C}$ ; then  $\gamma(\psi)(c) = \sum_{i=1}^n \psi(f_i)(c)c_i$ , and  $\delta(\zeta)(f)(c) = f(\zeta(c))$ . It is easy to check that  $\alpha$  is a ring isomorphism, while  $\delta$  is a ring anti-isomorphism.  $\square$

Let us show now how the duality theorem for finitely generated projective Hopf algebras can be derived from Theorem 1.1. We first need a description of the smash product as a dual of a coring. Before giving this result, let us remark that a coring interpretation of Theorem 0.2 ii) also realizes the smash product as the endomorphism ring of a coring.

**Proposition 1.2.** *Let  $H$  be a finitely generated projective Hopf algebra over the commutative ring  $k$ , and  $A$  a right  $H$ -comodule algebra. Let  $A \otimes H$  be the  $A$ -coring from example 0.5. Then there are ring isomorphisms*

$${}^*(A \otimes H) \simeq (A \otimes H)^* \simeq \#(H, A) \simeq A \# H^*.$$

*Proof.* Define

$$\gamma : (A \otimes H)^* \xrightarrow{\quad} \#(H, A) : \delta$$

by  $\gamma(\phi)(h) = \phi(1 \otimes S^{-1}(h))$ , and  $\delta(\zeta)(a \otimes h) = \zeta(a_{[1]}S(h))a_{[0]}$ . Since it is easy to see that  $\gamma(\phi)$  is  $k$ -linear and  $\delta(\zeta)$  is right  $A$ -linear, and that they are inverse one to each other, we check that they are ring isomorphisms:

$$\begin{aligned} (\gamma(\phi) \cdot \gamma(\psi))(h) &= \gamma(\phi)(\gamma(\psi)(h_{(2)})_{[1]}h_{(1)})\gamma(\psi)(h_{(2)})_{[0]} \\ &= \phi(1 \otimes S^{-1}(\psi(1 \otimes S^{-1}(h_{(2)})_{[1]}h_{(1)}))\psi(1 \otimes S^{-1}(h_{(2)})_{[0]}) \\ &= \phi(\psi(1 \otimes S^{-1}(h_{(2)})) \otimes S^{-1}(h_{(1)})) \\ &= \phi(\psi(1 \otimes S^{-1}(h_{(2)})(1 \otimes S^{-1}(h_{(1)}))) \\ &= \gamma(\phi * \psi)(h). \end{aligned}$$

This completes the proof for the middle isomorphism. Now define

$$\gamma' : {}^*(A \otimes H) \xrightarrow{\quad} \#(H, A) : \delta'$$

by  $\gamma'(\phi)(h) = \phi(1 \otimes h)$ , and  $\delta'(\zeta)(a \otimes h) = a\zeta(h)$ . It is clear that  $\gamma'(\phi)$  is  $k$ -linear and  $\delta'(\zeta)$  is left  $A$ -linear, and that they are inverse one to each other, so we check that they are ring isomorphisms. A short computation shows that  $\gamma'$  establishes a ring isomorphism  ${}^*(A \otimes H) \simeq \#(H^{op}, A^{op})^{op} \simeq (A^{op} \# H^{*cop})^{op}$ . Finally,

$$A \# H^* \simeq (A^{op} \# H^{*cop})^{op}, \quad a \# h^* \mapsto h_{(1)}^*(S^{-1}(a_{[1]}))a_{[0]} \# h_{(2)}^* S^{-1}$$

as rings, and the proof is complete.  $\square$

**Corollary 1.3.** *Let  $H$  be a finitely generated projective Hopf algebra over  $k$ , and  $A$  a right  $H$ -comodule algebra. Then  $(A \# H^*) \# H \simeq \text{End}(A \# H^*|_A)$  as rings.*

*Proof.* We have the following diagram of isomorphisms:

$$\begin{array}{ccc} \text{Hom}_{*C}({}^*C \otimes_A {}^*C, {}^*C) & \longrightarrow & \text{Hom}_A({}^*C, {}^*C) \\ \uparrow & & \nearrow \\ \text{Hom}_{*C}({}^*C \otimes H^*, {}^*C) & & \\ \uparrow & & \\ \text{Hom}_k(H^*, {}^*C) & & \end{array}$$

where  $C$  is the  $A$ -coring  $A \otimes H$  from Example 0.5, and the diagonal map is the composition of the two vertical maps and the horizontal one. By Proposition 1.2, the lower corner of the diagram is just  $(A \# H^*) \# H$ . Also

by Proposition 1.2,  ${}^*\mathcal{C}$  is a right  $H^*$ -comodule algebra. Let us denote this structure by  $c^* \mapsto c_{[0]}^* \otimes c_{[1]}^*$ . The horizontal isomorphism is the one from the first part of the proof of Theorem 1.1, the lower vertical one assigns to  $f \in \text{Hom}_k(H^*, {}^*\mathcal{C})$  the map sending  $c^* \otimes h^*$  to  $f(c_{[1]}^* h^* S) c_{[0]}^*$  (like  $\zeta$  from Proposition 1.2). The upper vertical isomorphism is induced by the map

$$\text{can} : {}^*\mathcal{C} \otimes_A {}^*\mathcal{C} \longrightarrow {}^*\mathcal{C} \otimes H^*, \quad c^* \otimes d^* \mapsto c^* d_{[0]}^* \otimes d_{[1]}^*.$$

Thus the diagonal map in the above diagram provides the isomorphism in the statement. In order to make sure that this is exactly the isomorphism from the duality theorem for finite Hopf algebras, we only need to show that it assigns to  $f$  the map sending  $c^*$  to  $f(c_{[1]}^*) c_{[0]}^*$ , which checks immediately.  $\square$

In view of the preceding Corollary, we get yet another possible point of view that trivializes the duality theorem for finite Hopf algebras: behind it we find just duality for finitely generated projective modules.

We also have an explanation for getting the bigger endomorphism ring of  $A$ , rather than  $A$  itself after taking the dual for the second time. Once we take the dual of a coring, we get a ring, so to get another coring that will enable us to take the second dual we need another construction, namely passing to the canonical coring.

## 2. A CORING VERSION OF SCHNEIDER'S ISOMORPHISM

Recall that if  $(\mathcal{C}, x)$  is an  $A$ -coring with fixed grouplike element  $x$ , and  $B = A^{co\mathcal{C}} = \{a \in A \mid \Delta(a) = a \otimes x\}$ , then  $(\mathcal{C}, x)$  is a Galois coring if the canonical coring morphism  $\text{can} : A \otimes_B A \longrightarrow \mathcal{C}$ ,  $\text{can}(a \otimes b) = axb$  is an isomorphism. We have the following coring version of Theorem 0.3:

**Theorem 2.1.** *Let  $(\mathcal{C}, x)$  be a Galois coring,  $M \in \mathcal{M}^{\mathcal{C}}$ , and  $N \in \mathcal{M}_A$ . Then*

$$\text{Hom}_A(M \otimes_A \mathcal{C}, N) \simeq \text{Hom}_B(M, N).$$

*Proof.*

$$\begin{aligned} \text{Hom}_A(M \otimes_A \mathcal{C}, N) &\simeq \text{Hom}_A(M \otimes_A A \otimes_B A, N) \\ &\simeq \text{Hom}_A(M \otimes_B A, N) \\ &\simeq \text{Hom}_B(M, \text{Hom}_A(A, N)) \\ &\quad (\text{by the adjunction property of the tensor product}) \\ &\simeq \text{Hom}_B(M, N). \end{aligned}$$

$\square$

Theorem 0.3 follows now immediately from Theorem 2.1: under the hypotheses of Theorem 0.3 (with  $H$  replaced by  $H^*$ ), take  $\mathcal{C} = A \otimes H$  as

in Example 0.5,  $x = 1 \otimes 1$ , and  $M = N \in \mathcal{M}^C$ . Then we have that  $H \otimes M \simeq M \otimes H$  as right  $A$ -modules, and therefore:

$$\begin{aligned} End_B(M) &\simeq Hom_A(M \otimes_A (A \otimes H), M) \\ &\simeq Hom_A(M \otimes H, M) \\ &\simeq Hom_A(H \otimes M, M) \\ &\simeq Hom_k(H, Hom_A(M, M)) \\ &\quad (\text{by the adjunction property of the tensor product}) \\ &\simeq End_A(M) \otimes H^*. \end{aligned}$$

We leave it to the reader to check that the isomorphism is the one from Theorem 0.3 (just note that the last isomorphism from above sends  $g$  to  $\sum_{i=1}^n g(h_i) \# h_i^* S^{-1}$ ).

In view of the above, we also obtain the surprising fact that what is behind the duality theorem for finite Hopf algebra and its refinements is just the hom-tensor adjunction property.

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