Coalgebras from Formulas

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Abstract

Nichols and Sweedler showed in [5] that generic formulas for sums may be used for producing examples of coalgebras. We adopt a slightly different point of view, and show that the reason why all these constructions work is the presence of certain representative functions on some (semi)group. In particular, the indeterminate in a polynomial ring is a primitive element because the identity function is representative.

Introduction

The title of this note is borrowed from the title of the second section of [5]. There it is explained how each generic addition formula naturally gives a formula for the action of the comultiplication in a coalgebra. Among the examples chosen in [5], this situation is probably best illustrated by the following two:
Let C be a k-space with basis $\{s, c\}$. We define $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$ by

$$\begin{aligned} \Delta(s) &= s \otimes c + c \otimes s \\ \Delta(c) &= c \otimes c - s \otimes s \\ \varepsilon(s) &= 0 \\ \varepsilon(c) &= 1. \end{aligned}$$

Then (C, Δ, ε) is a coalgebra called the trigonometric coalgebra.
Now let H be a k-vector space with basis $\{c_m \mid m \in \mathbf{N}\}$. Then H is a coalgebra with comultiplication Δ and counit ε defined by

$$\Delta(c_m) = \sum_{i=0,m} c_i \otimes c_{m-i}, \quad \varepsilon(c_m) = \delta_{0,m}.$$

This coalgebra is called the divided power coalgebra.
Identifying the "formulas" in the above examples is not hard: the formulas for sin and cos applied to a sum in the first example, and the binomial formula in the second one. Nevertheless, some more questions still need to be answered: Can one associate a coalgebra to any formula? If not, how can one characterize formulas leading to coalgebra structures? Do all formulas of that kind need to be formulas for sums? If not, are the formulas for sums special in any way?
The aim of this note is to answer all these questions. First, we remark that a formula will define a coalgebra structure precisely when it represents an equality showing that a certain function is a "representative function". The easiest, and perhaps the most striking example, is the coalgebra (or bialgebra) structure defined on the polynomial ring over an infinite field k, and it may be summarized as follows: the identity function from k to itself is a primitive representative function on the additive group of k. This in turn explains why "addition" plays a privileged role among the other operations: any polynomial function (or a function represented as the sum of a power series) is a representative function on the additive group of k, and the comultiplication applied to that function is in fact the function applied to a sum of variables.
Other examples include numerous coalgebra or bialgebra structures used in combinatorics.
All the above will be explained in the second section. Before doing this, we briefly recall in the first section the construction of the representative bialgebra of a semigroup (we include enough detail so that the exposition becomes self contained, but the reader is referred to [1] or [2] for unexplained notions or notation).

1 The representative bialgebra of a semigroup

Let k be a field, and G a monoid. Denote by kG the semigroup algebra (kG has basis G as a k-vector space and multiplication given by

$(ax)(by) = (ab)(xy)$ for $a, b \in k$, $x, y \in G$), and by $(kG)^*$ its linear dual, $(kG)^* = Hom_k(kG, k)$. Put

$$(kG)^\circ =$$

$$= \{f \in (kG)^* \mid \exists f_i, g_i \in (kG)^* \ : \ f(xy) = \sum f_i(x)g_i(y), \ \forall x, y \in kG\}$$

It is clear that $(kG)^\circ$ is a kG-subbimodule of $(kG)^*$ with respect to $\rightharpoonup$ and $\leftharpoonup$, given by

$$(x \rightharpoonup f)(y) = f(yx), \quad (f \leftharpoonup x)(y) = f(xy), \ \ \forall x, y \in kG, \ \ f \in (kG)^*.$$

If we take $f \in (kG)^\circ$, we can assume the f_i's and g_i's are linearly independent.
(This can be seen as follows: let n be the least positive integer for which there exist f_i, g_i with $f(xy) = \sum\limits_{i=1}^{n} f_i(x)g_i(y)$, for all $x, y \in kG$, with $(f_i)_{i=1,n}$ linearly independent. Then $(g_i)_{i=1,n}$ are also linearly independent, because if g_n is a linear combination of the others, say $g_n = \sum\limits_{i=1}^{n-1} \alpha_i g_i$, then $f(xy) = \sum\limits_{i=1}^{n-1} (f_i + \alpha_i f_n)(x)g_i(y)$, and $\{f_1 + \alpha_1 f_n, \ldots, f_{n-1} + \alpha_{n-1} f_n\}$ are linearly independent, contradicting the minimality of n).
Then there exist linearly independent $v_1, \ldots, v_n \in kG$ with $g_i(v_j) = 0$ for any $i, j = 1, \ldots, n$, $i \neq j$, while $g_i(v_i) \neq 0$. (This follows by induction on n. For $n = 1$ the result is clear. Let now $g_1, \ldots, g_{n+1}$ be linearly independent, and applying the induction hypothesis we find $v_1, \ldots, v_n \in kG$ satisfying the conditions for $g_1, \ldots, g_n$, and pick $v \in kG$ with $g_{n+1}(v) \neq \sum_{i=1,n} g_{n+1}(v_i)g_i(v)$. Then put $w_{n+1} = v - \sum_{i=1,n} (g_i(v_i))^{-1} v_i g_i(v)$, and $w_j = v_j - g_{n+1}(v_j)(g_{n+1}(w_{n+1}))^{-1} w_{n+1}$ for $j = 1, \ldots, n$. These $w_1, \ldots, w_{n+1}$ satisfy the required conditions).
Thus we obtain that $f_i \in kG \rightharpoonup f \subset (kG)^\circ$ and $g_i \in f \leftharpoonup kG \subset (kG)^\circ$. Hence we can define

$$\Delta : (kG)^\circ \longrightarrow (kG)^\circ \otimes (kG)^\circ,$$

$$\Delta(f) = \sum f_i \otimes g_i \quad \Leftrightarrow \quad f(xy) = \sum f_i(x)g_i(x), \ \ \forall x, y \in kG,$$

and

$$\varepsilon : (kG)^\circ \longrightarrow k, \quad \varepsilon(f) = f(1_G)$$

The above maps define a coalgebra structure on $(kG)^\circ$, which turns it into a bialgebra.
Also note that for all $f \in (kG)^\circ$, $kG \rightharpoonup f \leftharpoonup kG$ is a subcoalgebra of $(kG)^\circ$

(the smallest one containing f), and it is finite dimensional.
Now there exists an isomorphism of vector spaces

$$\phi : k^G \longrightarrow (kG)^* = Hom(kG, k), \;\; \phi(f)(\sum_i a_i x_i) = \sum_i a_i f(x_i).$$

Consequently, k^G becomes a kG-bimodule by transport of structures via ϕ:

$$(xf)(y) = f(yx), \quad (fx)(y) = f(xy), \;\; \forall x, y \in G, \;\; f \in k^G.$$

Definition 1.1 *If G is a monoid, we call*

$$R_k(G) := \phi^{-1}((kG)^\circ)$$

the representative bialgebra *of the monoid G.* ∎

Note that the bialgebra structure on $R_k(G)$ is also transported via ϕ. $R_k(G)$ is a kG-subbimodule of k^G, and consists of the functions (which are called representative) generating a finite dimensional kG-subbimodule (or, equivalently, a left or right kG-submodule). We have

$$R_k(G) = \{f \in k^G \mid \exists f_i, g_i \in k^G, \;\; f(xy) = \sum f_i(x) g_i(y) \;\; \forall x, y \in G\},$$

and the comultiplication map on $R_k(G)$ is given as follows: for $f \in R_k(G)$,

$$\Delta(f) = \sum f_i \otimes g_i \Leftrightarrow f_i, g_i \in k^G \text{ are such that } f(xy) = \sum f_i(x) g_i(y). \tag{1}$$

If G is a group, the $R_k(G)$ is a Hopf algebra with antipode $S(f)(x) = f(x^{-1})$.

Remarks 1.2 *1) The following is a partial explanation for the name of representative functions. Let G be a group, and $\rho : G \longrightarrow GL_n(k)$ a representation of G. Put $\rho(x) = (f_{ij}(x))_{i,j}$, and let $V(\rho)$ be the k-subspace of k^G spanned by the $\{f_{ij}\}_{i,j}$. Then $R_k(G) = \sum_\rho V(\rho)$, where ρ ranges over all finite dimensional representations of G. Indeed, let $f \in V(\rho)$, and $y \in G$. If $f = \sum a_{ij} f_{ij}$, then $(yf)(x) = \sum a_{ij} f_{ij}(xy) = \sum a_{ij} f_{ik}(x) g_{kj}(y)$, and thus $yf = \sum a_{ij} g_{kj} f_{ik} \in V(\rho)$. Similarly, $fy \in V(\rho)$, and so we have ($\supseteq$). For the reverse inclusion, let $f \in R_k(G)$. Then the left kG-submodule generated by f is finite dimensional, say with basis $\{f_1, \ldots, f_n\}$, and thus $xf_i = \sum g_{ij}(x) f_j$. Then $\rho : G \longrightarrow GL_n(k)$, $\rho(x) = (g_{ij}(x))_{i,j}$, is a representation of G, $V(\rho)$ is*

spanned by the g_{ij}'s, and we obtain that $f_i = \sum f_j(1_G)g_{ij} \in V(\rho)$, thus $f \in V(\rho)$.
2) Let $\theta : (kG)^* \otimes (kG)^* \longrightarrow (kG \otimes kG)^*$, $\theta(f \otimes g)(x \otimes y) = f(x)g(y)$. Then θ is an injective linear map, which is an isomorphism when G is finite. If we denote by $M : kG \otimes kG \longrightarrow kG$ the multiplication map, then $(kG)^\circ$ consists of those $f \in (kG)^*$ with the property that $M^*(f) \in Im(\theta)$. Consequently, $(kG)^\circ = (kG)^*$ when G is finite.
3) We give now a second explanation for the name of representative functions. Assume that G is a finite group. Then consider the representable functor from commutative k-algebras to groups, given by $F(R) = Alg_k((kG)^*, R)$ ($Alg_k((kG)^*, R)$ is a group under convolution, and the inverse of a map is the composition of that map with the antipode). For commutative k-algebras R without other idempotents than zero and one, we have that $F(R) \simeq G$, so in this case $(kG)^\circ = (kG)^*$ "almost represents" the functor associating the group G to any commutative k-algebra R. ∎

2 Representative Functions and Coalgebra Structures

We start this section by looking at some examples of representative functions:
1) Let G be a monoid. Then $f \in R_k(G)$ is a grouplike element if and only if $\Delta(f) = f \otimes f$ and $\varepsilon(f) = 1$, which will happen if and only if, by (1), $f(xy) = f(x)f(y)$ and $f(1_G) = 1$, meaning that f is a monoid morphism from G to $(k, \cdot)$.
An example is the exponential function $\exp : \mathbf{R} \longrightarrow \mathbf{R}$, because $e^{x+y} = e^x e^y$, hence $\Delta(\exp) = \exp \otimes \exp$, and the subspace spanned by exp in $R_{\mathbf{R}}((\mathbf{R}, +))$ is a one-dimensional subcoalgebra.
Another example is $\mathbf{1}$, the constant function taking the value 1: we have $\mathbf{1}(x + y) = 1 = \mathbf{1}(x)\mathbf{1}(y)$.
2) If G is a monoid, then $f \in R_k(G)$ is primitive if and only if $\Delta(f) = f \otimes \mathbf{1} + \mathbf{1} \otimes f$, where $\mathbf{1}$ is the constant function taking the value 1. By (1), this will happen if and only if $f(xy) = f(x)\mathbf{1}(y) + \mathbf{1}(x)f(y) = f(x) + f(y)$, i.e. if and only if f is a semigroup morphism from G to $(k, +)$.
An example is the logarithmic function $\lg : (0, \infty) \longrightarrow \mathbf{R}$, because $\lg(xy) = \lg(x) + \lg(y)$, hence $\Delta(\lg) = \lg \otimes \mathbf{1} + \mathbf{1} \otimes \lg$, and thus $\mathbf{1}$ and lg span a two-dimensional subcoalgebra of $R_{\mathbf{R}}((0, \infty), \cdot))$.

3) Let $d_n : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $d_n(x) = \frac{x^n}{n!}$. Since $d_n(x+y) = \sum_i d_i(x)d_{n-i}(y)$ (by the binomial formula), it follows that the d_n's are representative functions on the group $(\mathbf{R}, +)$, and the subspace they span is a subcoalgebra of $R_{\mathbf{R}}((\mathbf{R}, +))$, isomorphic to the divided power coalgebra from the introduction. This explains the name of this coalgebra.
The binomial formula for $n = 0$ just says that $\mathbf{1}$, the constant function 1, is a grouplike, because $(x+y)^0 = \mathbf{1}(x+y) = 1 = \mathbf{1}(x)\mathbf{1}(y)$.
For $n = 1$, it shows that the identity function is primitive: $x+y = x+y$ may be seen as $Id(x+y) = Id(x) + Id(y) = Id(x)\mathbf{1}(y) + \mathbf{1}(x)Id(y)$.
In general, if we replace $\mathbf{R}$ by any infinite field k (so that polynomial functions on k correspond one-to-one with polynomials in $k[X]$), this explains the definition of the comultiplication defined on the polynomial ring $k[X]$: the indeterminate X corresponds to the identity function, which is primitive. Consequently, in this case all polynomial functions are representative.

Divided power coalgebras are behind other examples of coalgebras appearing in combinatorics. We briefly list three of their possible generalizations (see [3] for more details), along with some other appearances of representative functions.
4) Incidence coalgebras for partially ordered sets may be viewed as generalizations of the divided powers coalgebra, since the latter is just the standard reduced incidence coalgebra spanned by segments of nonnegative integers under natural ordering.
5) Divided powers coalgebras can also be extended in a different direction, by generalizing the binomial coefficients as follows: Let G be a commutative semigroup (written additively). Then *section coefficients* on G are a mapping $(i, j, k) \mapsto (i \mid j, k) \in \mathbf{Z}$ such that for any i the number of ordered pairs j, k such that $(i \mid j, k) \neq 0$ is finite, and

$$\sum_k (i \mid j, k)(k \mid p, q) = \sum_s (i \mid s, q)(s \mid j, p) \tag{2}$$

The section coefficients are called *bisection coefficients* if

$$(i + j \mid p, q) = \sum_{p_1+p_2=p, q_1+q_2=q} (i \mid p_1, q_1)(j \mid p_2, q_2) \tag{3}$$

Therefore, the bisection coefficients may be viewed as representative functions on G (over a field k of characteristic zero). To associate a coalgebra C to G, associate an x_i to each $i \in G$, take the span C of all the x_i's, and define

$$\Delta(x_i) = \sum_{j,k} (i \mid j,k) x_j \otimes x_k.$$

Also put $\varepsilon(x_i) = \delta_{i,0}$, where $0 \in G$ is unique such that $(i \mid 0, j) = (i \mid j, 0) = \delta_{i,j}$. Then the coassociativity of Δ follows from (2), while putting $x_j x_j = x_{i+j}$ turns C into a bialgebra because of (3). This is an illustration of how useful is the fact that being "representative" ensures both coassociativity of the comultiplication and compatibility with the multiplication.

6) Another possible extension of divided powers coalgebras uses polynomial sequences of binomial type as a replacement for the sequence d_n from 3). By definition, these are representative functions on the additive group of k (see [3, (5.4)]): the ploynomial sequence $p_n(x)$ is said to be of *binomial type* if deg $p_n = n$ for all n, and

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y).$$

Similar examples of representative functions are provided by polynomial sequences of Boolean type (see [3, (5.8)]): a polynomial sequence indexed by the finite subsets of a set $\{p_A(x)\}$ is said to be of *Boolean type* if

$$p_A(x+y) = \sum_{A_1 + A_2 = A} p_{A_1}(x) p_{A_2}(y).$$

The motivating examples behind them are chromatic polynomials of graphs (see [3, (5.9)]), which are also examples of representative functions.

We now go back to the first example in the introduction, and give an explanation for the name "trigonometric coalgebra". The functions sin and cos : $\mathbf{R} \longrightarrow \mathbf{R}$ satisfy the equalities

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

and

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

These equalities are both of the type (1), and show that sin and cos are representative functions on the group $(\mathbf{R}, +)$. The subspace generated by them in the space of the real functions is then a subcoalgebra of $R_{\mathbf{R}}((\mathbf{R}, +))$, isomorphic to the trigonometric coalgebra.

As in Remark 1.2 3), the fact that sin and cos are representative functions is also suitable for a different interpretation. Instead of taking the subspace of $R_{\mathbf{R}}((\mathbf{R},+))$ spanned by sin and cos, take the subalgebra generated by them, $\mathbf{R}[\sin,\cos]$, factor it by the ideal generated by $\sin^2+\cos^2-1$, and denote the quotient algebra by H. Then H is a commutative Hopf algebra, with comultiplication defined on sin and cos as above. Moreover, as shown in [6, Section 2.2], the Hopf algebra H represents (as in "representable functor represented by the commutative $\mathbf{R}$-algebra H") the affine group scheme $\mathcal{C}:\mathbf{R}-\mathbf{Alg_c}\longrightarrow\mathbf{Gr}$, defined by

$$\mathcal{C}(R)=\{(a,b)\in R\times R\mid a^2+b^2=1\},$$

on which the group structure is defined by

$$(a,b)\cdot(c,d)=(ac-bd,ad+bc).$$

Finally, we have a brief look at what makes addition special among other operations (the fact that addition is not the only operation providing formulas leading to coalgebras is illustrated in 2) above).
Let k be an infinite field. Then we have the k-algebra isomorphism

$$k[X,Y]\longrightarrow k[X]\otimes k[X],\quad X\mapsto X\otimes 1,\;\; Y\mapsto 1\otimes X.$$

Let P be a polynomial function. Since P is representative, there exist (polynomial) functions F_i and G_i, $i=1,\dots,n$ such that

$$P(x+y)=\sum_{i=1}^{n}F_i(x)G_i(y). \tag{4}$$

(Note that (4) may also be derived simply by applying the binomial theorem several times and collecting like terms.) Consequently, by (1) we get that $\Delta(P)=\sum F_i\otimes G_i$. The same thing may be obtained directly from (4) by applying Δ to P and using the fact that Δ is multiplicative:

$$\begin{aligned}\Delta(P(x)) &= P(\Delta(x))\\ &= P(x\otimes 1+1\otimes x)\\ &= P(x+y)\\ &= \sum_{i=1}^{n}F_i(x)G_i(y)\\ &= \sum_{i=1}^{n}F_i(x\otimes 1)G_i(1\otimes x)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} (F_i(x) \otimes 1)(1 \otimes G_i(x)) \\
&= \sum_{i=1}^{n} F_i(x) \otimes G_i(x).
\end{aligned}
$$

This way of obtaining a formula for Δ from the formula (4) without mentioning representative functions or (1) was used in [5].

We end by remarking that the point of view exhibiting various representative functions behind coalgebra structures may be expanded by looking at Hopf algebras acting on algebras (or fields), because the measuring condition is also a way of saying that acting by an element of the Hopf algebra is a representative function (i.e. it is a formula of type (1). In this way, "acting as homomorphisms" produces grouplikes, while "acting as derivations" produces primitives.

All of the above come as no surprise: since it is well known that any coalgebra may be viewed as a subcoalgebra of the representative coalgebra of the multiplicative monoid of its linear dual [2, Exercise 1.5.15], wherever there is a coalgebra there are also representative functions around.

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