# BALANCED BILINEAR FORMS FOR CORINGS

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*Dedicated to Robert Wisbauer on the occasion of his 65th birthday*

Abstract. We review the role that balanced bilinear forms play in the definitions of properties of corings and suggest a definition for a coring to be symmetric.

## 1. Introduction

Let $k$ be a commutative ring and $A$ a finite dimensional $k$-algebra. An element $e$ in $A \otimes_k A$ is called $A$-central, or a Casimir element [3, Section 1.3], if $ae = ea$ for all $a \in A$. Various properties of $A$ are defined in terms of $A$-central elements. For example, $A$ is $k$-separable if and only if $A$ has an $A$-central element $e$ such that $\pi(e) = 1$ where $\pi : A \otimes_k A \to A$ is the usual map $\pi(a \otimes_k b) = ab$. $A$ is a Frobenius $k$-algebra if and only if there exists an $A$-central element $e$ and a map $\epsilon \in A^*$ such that $(\epsilon \otimes_k A)(e) = (A \otimes_k \epsilon)(e) = 1$. Equivalently, $A$ is Frobenius if and only if there is a nondegenerate bilinear map $B : A \times A \to k$ such that $B(xy, z) = B(x, yz)$. The algebra $A$ is symmetric if $A$ is Frobenius and $B(x, y) = B(y, x)$.

For $C$ a coalgebra over $k$, various analogous properties of $C$ may be defined in terms of balanced bilinear forms from $C \otimes_k C$ to $k$, generalizing the idea of $A$-central element. A $C^*$-balanced form is a $k$-bilinear form $B$ from $C \otimes_k C$ to $k$ such that $B(c \leftharpoonup c^*, d) = B(c, c^* \rightharpoonup d)$ for all $c, d \in C$, $c^* \in C^*$ with the usual actions of $C^*$ on $C$. For $k$ a field, the idea of a symmetric coalgebra was recently defined in [4], namely that a $k$-coalgebra $C$ is symmetric if and only if there is a nondegenerate symmetric balanced bilinear form $B$ from $C \otimes_k C$ to $k$.

For $\mathcal{C}$ an $A$-coring, where $A$ is not necessarily commutative, the situation is complicated by the presence of left and right duals. However, the idea of balanced bilinear forms from $\mathcal{C} \otimes_A \mathcal{C}$ to $A$ still makes sense and is used to define various properties of corings analogous to those for coalgebras. We recall some of these properties, and, in the last section, suggest a working definition for the notion of a symmetric coring along with some examples.

We will work over a commutative ring $k$ and all maps are assumed to be $k$-linear. Throughout this paper, $A$ will denote a not necessarily commutative $k$-algebra. We will use the Sweedler summation notation for comultiplication, but omitting the summation sign. For background on coalgebras over a field we refer the reader to [5]. The first chapter of [2] contains basics on coalgebras over a commutative ring. The identity map on a $k$-module $X$ is denoted simply as $X$.

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## 2. Corings

Recall that for $A$ a not necessarily commutative ring, an $A$-coring $\mathcal{C}$ is defined to be a coalgebra in the monoidal category of $(A, A)$-bimodules, $({}_A\mathcal{M}_A, \otimes_A, A)$. More precisely, $\mathcal{C}$ is an $(A, A)$-bimodule, together with $(A, A)$-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ such that $\Delta_{\mathcal{C}}$ is co-associative and the compatibility conditions $(\epsilon_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}}(c) = c$ and $(\mathcal{C} \otimes_A \epsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}(c) = c$, for all $c \in \mathcal{C}$, hold.

For definitions and details about corings, we refer the reader either to [2] or, for the original definition, to [9]. We will normally write $(A, A)$-bimodule actions on a module $M \in {}_A\mathcal{M}_A$ by juxtaposition, i.e. we write $amb$ for the left action of $a$ and the right action of $b$ on $m$.

If $A = k$, then we recover the definition of a coalgebra over the commutative ring $k$. Simple examples of corings include the following.

**Examples 2.1.** (i) **Trivial coring.** *For $A$ a ring, let $\mathcal{C} = A$ itself and define $\Delta_{\mathcal{C}}(a) = a \otimes_A 1 = 1 \otimes_A a$ and $\epsilon_{\mathcal{C}}(a) = a$.*

(ii) **Matrix coring.** (See [2, 17.7].) *For $A$ a ring, let $\mathcal{C} = M_n^c(A)$, $n$ by $n$ matrices over $A$ with $A$-basis $e_{ij}$, $1 \le i, j \le n$ and $ae_{ij} = e_{ij}a$. Then $(\mathcal{C}, \Delta_{\mathcal{C}}, \epsilon_{\mathcal{C}})$ is an $A$-coring where $\Delta_{\mathcal{C}}(e_{ij}) = \sum_{k=1}^n e_{ik} \otimes_A e_{kj}$ and $\epsilon_{\mathcal{C}}(e_{ij}) = \delta_{i,j}$, and the maps are extended by $A$-linearity.*

(iii) **An entwining structure example.** *Let $k$ be a field, let $H$ be a $k$-Hopf algebra, and $A$ a right $H$-comodule algebra via $a \mapsto a_{[0]} \otimes_k a_{[1]}$. Then $A \otimes_k H$ becomes an $A$-coring as follows: the left $A$-module structure is given by multiplication on the first component, and the right $A$-module structure is given by $(a \otimes_k h)b = ab_{[0]} \otimes_k hb_{[1]}$. The comultiplication is $\Delta : A \otimes_k H \longrightarrow (A \otimes_k H) \otimes_A (A \otimes_k H) \simeq A \otimes_k H \otimes_k H$, $\Delta(a \otimes_k h) = (a \otimes_k h_{(1)}) \otimes_A (1_A \otimes_k h_{(2)})$, and the counit is $\epsilon : A \otimes_k H \longrightarrow A$, $\epsilon(a \otimes_k h) = \epsilon(h)a$.*

(iv) **Opposite coring.** *Let $A^o$ be the opposite algebra of $A$. If $M$ is an $(A, A)$-bimodule, then $M$ is also an $(A^o, A^o)$-bimodule, denoted $M^o$, as usual, via $a^o m^o b^o = (bma)^o$. The twist map $\tau : M \otimes_A M \to M^o \otimes_{A^o} M^o$ defined by $\tau(m \otimes_A n) = n^o \otimes_{A^o} m^o$ is a $k$-module isomorphism. The opposite coring, denoted $\mathcal{C}^o$, is defined* [6, 1.7] *to be the $A^o$-coring $(\mathcal{C}, \triangle^o_{\mathcal{C}}, \epsilon_{\mathcal{C}})$, where comultiplication $\Delta^o_{\mathcal{C}}(c^o) = \tau \circ \triangle_{\mathcal{C}}(c) = (c_2)^o \otimes_{A^o} (c_1)^o$.*

If $A = k$, and $C$ is a $k$-coalgebra, then the dual $C^*$ is an algebra via the convolution map. However, for $\mathcal{C}$ a coring, there are right and left dual rings associated with $\mathcal{C}$, denoted $\mathcal{C}^*$ and ${}^*\mathcal{C}$.

Following [2, 17.8], we write $\mathcal{C}^* := \mathrm{Hom}_A(\mathcal{C}, A)$, the right $A$-module homomorphisms from $\mathcal{C}$ to $A$. $\mathcal{C}^*$ has a ring structure with associative multiplication $*^r$ given by $f *^r g(c) = g(f(c_1)c_2)$ and unit $\epsilon_{\mathcal{C}}$. There is a ring morphism from $A^o$ to $\mathcal{C}^*$ by $a^o \mapsto \epsilon_{\mathcal{C}}(a-)$. Thus $\mathcal{C}^*$ has left $A^o$-action via $(\epsilon_{\mathcal{C}}(b-) *^r c^*)(c) = c^*(bc)$ and, similarly, a right $A^o$-action. Then $\mathcal{C}^*$ is an $(A, A)$-bimodule via $(ac^*b)(c) = ac^*(bc)$.

Similarly, ${}^*\mathcal{C} := {}_A\mathrm{Hom}(\mathcal{C}, A)$, the left $A$-module homomorphisms from $\mathcal{C}$ to $A$. ${}^*\mathcal{C}$ has a ring structure with associative multiplication $*^l$ given by $f *^l g(c) = f(c_1 g(c_2))$ and unit $\epsilon_{\mathcal{C}}$. There is a ring morphism from $A^o$ to ${}^*\mathcal{C}$ given by $a \mapsto \epsilon_{\mathcal{C}}(-a)$. Then ${}^*\mathcal{C}$ is an $(A, A)$-bimodule via $(ac^*b)(c) = c^*(ca)b$.

Note that convolution is not well defined on either $\mathcal{C}^*$ or ${}^*\mathcal{C}$. The problem is that $\Delta_{\mathcal{C}}$ maps to $\mathcal{C} \otimes_A \mathcal{C}$ and $A$-linearity may fail.

Again, using notation from [2], we denote ${}^*\mathcal{C}^* := {}_A\mathrm{Hom}_A(\mathcal{C}, A) = {}^*\mathcal{C} \cap \mathcal{C}^*$ to be the set of $(A, A)$-bimodule maps from $\mathcal{C}$ to $A$. On ${}^*\mathcal{C}^*$ the associative multiplications

$*^l$ and $*^r$ both equal the convolution multiplication, so that ${}^*\mathcal{C}^*$ with the convolution multiplication $f * g(c) = f(c_1)g(c_2)$ is an associative ring with unit $\epsilon_{\mathcal{C}}$.

**Remark 2.2.** *${}^*\mathcal{C}^*$ is not a left or a right $A$-module under either the $A$-module structures of ${}^*\mathcal{C}$ or of $\mathcal{C}^*$. For suppose $f$ lies in ${}^*\mathcal{C}^*$ and we attempt to define $af$ by $(af)(c) = f(ca)$ (the left $A$-module structure on ${}^*\mathcal{C}$). Now $af$ may not lie in ${}^*\mathcal{C}^* \subset \mathcal{C}^*$ since $(af)(cb) = f(cba) = f(c)ba$ which is not, in general, equal to $(af)(c)b = f(c)ab$. Similarly, if we let $(af)(c) = af(c)$, then $af$ may not lie in ${}^*\mathcal{C}^*$. However, if $a \in Z(A)$, the centre of $A$, then the definitions above of left module structure agree and $af \in {}^*\mathcal{C}^*$, so that ${}^*\mathcal{C}^*$ is a left $Z(A)$-module. Similarly, the right module structures on ${}^*\mathcal{C}$ and $\mathcal{C}^*$ also may not induce right $A$-module structures on ${}^*\mathcal{C}^*$ but do induce a right $Z(A)$-module structure.*

For $C$ a coalgebra over a commutative ring $k$, it is well known that $C$ is a $(C^*, C^*)$-bimodule. For corings the situation is somewhat different.

The coring $\mathcal{C}$ is a right $\mathcal{C}^*$-module via $c \leftharpoonup c^* = c^*(c_1)c_2$ and the right $A$-action on $\mathcal{C}$ commutes with the right $\mathcal{C}^*$-action. However, $\mathcal{C}$ is not an $(A, \mathcal{C}^*)$-bimodule in general since, for $c^* \in \mathcal{C}^*$, we need not have equality of $c^*(ac_1)c_2$ and $ac^*(c_1)c_2$. Similarly, there is a left ${}^*\mathcal{C}$-action on $\mathcal{C}$ which commutes with the left $A$-action given by $c^* \rightharpoonup c = c_1c^*(c_2)$. It follows from the coassociativity of $\Delta_{\mathcal{C}}$ and the fact that $\Delta_{\mathcal{C}}$ is an $(A, A)$-bimodule map, that $\mathcal{C}$ is a $({}^*\mathcal{C}, \mathcal{C}^*)$-bimodule with the above actions.

## 3. Balanced bilinear forms for corings

Recall that $M \in {}_A\mathcal{M}$ is locally projective as a left $A$-module if and only if for any finite set $S$ of elements of $M$, there exist $x_1, \ldots, x_n \in M, f_1, \ldots, f_n \in {}_A\mathrm{Hom}(M, A) = {}^*M$ such that $m = \sum_{i=1}^n f_i(m)x_i$, for any $m \in S$. Any object of $\mathcal{M}^{\mathcal{C}}$, the category of right $\mathcal{C}$-comodules, can be viewed as an object of ${}_{{}^*\mathcal{C}}\mathcal{M}$, the category of left ${}^*\mathcal{C}$-modules, and if $\mathcal{C}$ is locally projective, then $\mathcal{M}^{\mathcal{C}}$ is a full subcategory of ${}_{{}^*\mathcal{C}}\mathcal{M}$. Moreover, the rational functor $\mathrm{Rat}^{\mathcal{C}} : {}_{{}^*\mathcal{C}}\mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ is defined. Similar statements hold for right local projectivity.

Suppose that $\mathcal{C}$ is locally projective as a left $A$-module. Recall from [2, Section 20], that if $M \in \mathcal{M}_A$ is a left ${}^*\mathcal{C}$-module, $\mathrm{Rat}^{\mathcal{C}}(M)$, the rational submodule of $M$, may be defined as the set of rational elements of $M$, where $m \in M$ is called rational if there exists $\sum_{i=1}^n m_i \otimes_A c_i \in M \otimes_A \mathcal{C}$ such that $\phi \cdot m = \sum_i m_i\phi(c_i)$, for all $\phi$ in ${}^*\mathcal{C}$. By the locally projective condition on $\mathcal{C}$, $\sum_i m_i \otimes_A c_i$ is uniquely determined and so these elements define a right $\mathcal{C}$-comodule structure on $\mathrm{Rat}^{\mathcal{C}}(M)$. For $M = {}^*\mathcal{C}$ we have that $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ is an ideal of ${}^*\mathcal{C}$ (and thus also an $A$-subbimodule of ${}^*\mathcal{C}$).

For $\mathcal{C}$ locally projective as a right $A$-module, similar statements hold. Here the rational functor is denoted ${}^{\mathcal{C}}\mathrm{Rat} : \ \mathcal{M}_{\mathcal{C}^*} \to {}^{\mathcal{C}}\mathcal{M}$, and ${}^{\mathcal{C}}\mathrm{Rat}(\mathcal{C}^*)$ is an ideal of $\mathcal{C}^*$.

With the $(A, A)$-bimodule structure on $\mathcal{C} \otimes_A \mathcal{C}$ given by $a(c \otimes_A c')b = ac \otimes_A c'b$, the set of balanced bilinear forms on $\mathcal{C}$ is defined as follows.

**Definition 3.1.** *The set of balanced bilinear forms on $\mathcal{C}$, denoted $\mathrm{bbf}(\mathcal{C})$, is defined to be the set of $\sigma \in {}_A\mathrm{Hom}_A(\mathcal{C} \otimes_A \mathcal{C}, A)$ such that*

$$(\sigma \otimes_A \mathcal{C}) \circ (\mathcal{C} \otimes_A \Delta_{\mathcal{C}}) = (\mathcal{C} \otimes_A \sigma) \circ (\Delta_{\mathcal{C}} \otimes_A \mathcal{C})$$

*or, equivalently,*

$$\sigma(c \otimes_A d_1)d_2 = c_1\sigma(c_2 \otimes_A d), \ \textit{for all } c, d \in \mathcal{C}. \tag{3.1}$$

By [2, 6.4], if $\mathcal{C} = C$ is a locally projective $k$-coalgebra, then (3.1) is equivalent to the defining relation for $C^*$-balanced forms $B$ given in the introduction.

As usual, $\sigma \in \mathrm{bbf}(\mathcal{C})$ is called right nondegenerate if $\sigma(c \otimes_A \mathcal{C}) = 0$ implies $c = 0$, and left nondegenerate if $\sigma(\mathcal{C} \otimes_A c) = 0$ implies $c = 0$.

**Examples 3.2.** (i) *For $\mathcal{C} = A$, the trivial coring of Examples 2.1, let $\sigma = \epsilon_{\mathcal{C}} \otimes_A \epsilon_{\mathcal{C}}$. It is easy to see that $\sigma \in \mathrm{bbf}(\mathcal{C})$ and is nondegenerate.*

(ii) *For $\mathcal{C}$ the $n$ by $n$ matrix coring of Examples 2.1, define $\sigma$ to be the $A$-linear map from $\mathcal{C} \otimes_A \mathcal{C}$ to $k$ defined on generators by $\sigma(e_{ij} \otimes_A e_{kl}) = \delta_{i,l}\delta_{j,k}$. Again, it is easily checked that $\sigma \in \mathrm{bbf}(\mathcal{C})$ and is nondegenerate.*

(iii) *Let $\mathcal{C}$ be the coring $A \otimes_k H$ from Examples 2.1 and suppose that $H$ has a left integral $t$ in $H^*$. It is well known that $B(h, g) = t(hS(g))$ where $S$ is the antipode of $H$, is a balanced form on $H$ since $h_1 t(h_2 S(g)) = h_1 S(g_2) g_3 t(h_2 S(g_1)) = t(hS(g_1)) g_2$ for all $g, h \in H$. Now define $\sigma : \mathcal{C} \otimes_A \mathcal{C} \to A$ by*

$$\sigma((a \otimes_k h) \otimes_A (b \otimes_k g)) = ab_{[0]} t(hb_{[1]} S(g)).$$

*It is easy to check that $\sigma$ is $A$-bilinear and the balanced property comes from that of $B$.*

(iv) *Let $\mathrm{bbf}(\mathcal{C}^o)$ denote the set of balanced bilinear forms for the $A^o$-coring $\mathcal{C}^o$. For $\sigma \in \mathrm{bbf}(\mathcal{C})$, define $\sigma^o : \mathcal{C}^o \otimes_{A^o} \mathcal{C}^o \to A^o$ by $\sigma^o(c^o \otimes_{A^o} d^o) = (\sigma \circ \tau(c^o \otimes_{A^o} d^o))^o = (\sigma(d \otimes_A c))^o$. It is easily checked that $\sigma^o \in \mathrm{bbf}(\mathcal{C}^o)$.*

Now, let $\sigma \in \mathrm{bbf}(\mathcal{C})$. Since $\sigma$ is a left and right $A$-module map, $\sigma$ induces a well defined left $A$-module map $\sigma^r : \mathcal{C} \to \mathcal{C}^*$, given by $\sigma^r(c)(d) = \sigma(c \otimes_A d)$. It is easily checked that $\sigma^r$ is also a right $A$-module map and a right $\mathcal{C}^*$-module map. Similarly, $\sigma^l : \mathcal{C} \to {}^*\mathcal{C}$ defined by $\sigma^l(d)(c) = \sigma(c \otimes_A d)$ is well defined, a left ${}^*\mathcal{C}$-module and an $(A, A)$-bimodule map.

The next lemma is a straightforward generalization to corings of well-known facts for coalgebras. Statements (i) and (ii) can be found in [8, Proposition 1] for coalgebras over a field, and in [2, Section 6.6] for locally projective coalgebras over a commutative ring.

**Lemma 3.3.** *We have the following bijective correspondences.*

(i) *Let $\mathcal{C}$ be an $A$-coring which is locally projective as a right and as a left $A$-module. There is a bijective correspondence between $\mathrm{bbf}(\mathcal{C})$ and the set of $(A, A)$-bimodule, right $\mathcal{C}^*$-module maps from $\mathcal{C}$ to ${}^{\mathcal{C}}\mathrm{Rat}(\mathcal{C}^*)$. Under this correspondence, right nondegenerate forms correspond to monomorphisms from $\mathcal{C}$ to ${}^{\mathcal{C}}\mathrm{Rat}(\mathcal{C}^*)$.*

(ii) *For $\mathcal{C}$ an $A$-coring which is locally projective as a right and as a left $A$-module, there is a bijective correspondence between $\mathrm{bbf}(\mathcal{C})$ and the set of $(A, A)$-bimodule, left ${}^*\mathcal{C}$-module maps from $\mathcal{C}$ to $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$. Under this correspondence, left nondegenerate forms correspond to monomorphisms from $\mathcal{C}$ to $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$.*

(iii) *For any $A$-coring $\mathcal{C}$, (not necessarily locally projective), there is a bijective correspondence between $\mathrm{bbf}(\mathcal{C})$ and the set of $(\mathcal{C}, \mathcal{C})$-bicomodule maps from $\mathcal{C} \otimes_A \mathcal{C}$ to $\mathcal{C}$, where $\mathcal{C} \otimes_A \mathcal{C}$ is a left (right) $\mathcal{C}$-comodule via $\triangle_{\mathcal{C}} \otimes_A \mathcal{C}$ ($\mathcal{C} \otimes_A \triangle_{\mathcal{C}}$, respectively).*

*Proof.* (i) First we show that there is a bijective correspondence between $\mathrm{bbf}(\mathcal{C})$ and ${}_A\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}^*) \cap \mathrm{Hom}_{\mathcal{C}^*}(\mathcal{C}, \mathcal{C}^*)$. Suppose that $\varphi : \mathcal{C} \to \mathcal{C}^*$ is an $(A, A)$-bimodule, right $\mathcal{C}^*$-module map. Define $\sigma := \sigma_\varphi : \mathcal{C} \otimes_A \mathcal{C} \to A$ by $\sigma_\varphi(c \otimes_A d) = \varphi(c)(d)$, for all $c, d \in \mathcal{C}$. We must show that $\sigma_\varphi$ is a well-defined $(A, A)$-bimodule map and is balanced.

Since $\varphi$ is a right $A$-module map, we have

$$\sigma(ca \otimes_A d) = \varphi(ca)(d) = (\varphi(c)a)(d) = \varphi(c)(ad) = \sigma(c \otimes_A ad),$$

and so $\sigma$ is well defined. Next, note that for $a, a' \in A$, since $\varphi(ac) = a\varphi(c)$ and $\varphi(ac) \in \mathcal{C}^*$, then

$$\sigma(ac \otimes_A da') = \varphi(ac)(da') = a(\varphi(c)(d))a',$$

so that $\sigma$ is an $(A, A)$-bimodule map. Finally, to see that $\sigma \in \mathrm{bbf}(\mathcal{C})$, note that for $c \in \mathcal{C}$, $c^* \in \mathcal{C}^*$, since $\varphi(c \leftharpoonup c^*) = \varphi(c) *^r c^*$, then

$$c^*(c_1\sigma(c_2 \otimes_A d)) = c^*(\sigma(c \otimes_A d_1)d_2),$$

for all $d \in \mathcal{C}$. Hence $c_1\sigma(c_2 \otimes_A d)$ and $\sigma(c \otimes_A d_1)d_2$ have the same image under every $c^* \in \mathcal{C}^*$ and thus, since $\mathcal{C}$ is locally projective as an $A$-module, they are equal.

Conversely, given $\sigma \in \mathrm{bbf}(\mathcal{C})$, define the right $\mathcal{C}^*$-module map $\sigma^r$ as above. It is easy to check that $\sigma \mapsto \sigma^r$ and $\varphi \mapsto \sigma_\varphi$ define inverse bijections between $\mathrm{bbf}(\mathcal{C})$ and ${}_A\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}^*) \cap \mathrm{Hom}_{\mathcal{C}^*}(\mathcal{C}, \mathcal{C}^*)$. Clearly, right nondegenerate forms correspond to monomorphisms.

Now we show that any map in ${}_A\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}^*) \cap \mathrm{Hom}_{\mathcal{C}^*}(\mathcal{C}, \mathcal{C}^*)$ has its image in ${}^{\mathcal{C}}\mathrm{Rat}(\mathcal{C}^*)$ so that ${}_A\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}^*) \cap \mathrm{Hom}_{\mathcal{C}^*}(\mathcal{C}, \mathcal{C}^*)$ is the set of $(A, A)$ bimodule, right $\mathcal{C}^*$-module maps from $\mathcal{C}$ to ${}^{\mathcal{C}}\mathrm{Rat}(\mathcal{C}^*)$. Let $\sigma \in \mathrm{bbf}(\mathcal{C})$ and $\varphi = \sigma^r$; we show that every element $\varphi(c)$ is rational, i.e., for each $c \in \mathcal{C}$, there exists $\sum_i c_i \otimes_k x_i \in \mathcal{C} \otimes_k \mathcal{C}^*$ such that $\varphi(c) *^r c^* = \sum_i c^*(c_i)x_i$, for all $c^* \in \mathcal{C}^*$. Let $\sum_i c_i \otimes_k x_i = c_1 \otimes_k \varphi(c_2)$. We have that for any $c^* \in \mathcal{C}^*$ and $c, d \in \mathcal{C}$,

$$\begin{aligned} (\varphi(c) *^r c^*)(d) &= c^*(\varphi(c)(d_1)d_2) = c^*(\sigma(c \otimes_A d_1)d_2) \\ &= c^*(c_1\sigma(c_2 \otimes_A d)) = c^*(c_1)\varphi(c_2)(d) \text{ (since } c^* \in \mathcal{C}^*), \end{aligned}$$

so that $\varphi(c) *^r c^* = c^*(c_1)\varphi(c_2)$, showing that $\varphi(c)$ is rational.

(ii) For $\gamma : \mathcal{C} \to {}^*\mathcal{C}$ an $(A, A)$-bimodule, left ${}^*\mathcal{C}$-module map, define $\sigma_\gamma : \mathcal{C} \otimes_A \mathcal{C} \to A$ by $\sigma_\gamma(c \otimes_A d) = \gamma(d)(c)$. Conversely, for $\sigma \in \mathrm{bbf}(\mathcal{C})$, define $\sigma^l$ as above. The proof that these provide inverse bijections and that for $\gamma \in {}_A\mathrm{Hom}_A(\mathcal{C}, {}^*\mathcal{C}) \cap {}_{{}^*\mathcal{C}}\mathrm{Hom}(\mathcal{C}, {}^*\mathcal{C})$ then the image of $\gamma$ lies in $\mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C})$ is analogous to the proof of (i).

(iii) Take $\sigma \in \mathrm{bbf}(\mathcal{C})$ and define $m_\sigma : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$ by $m_\sigma(c \otimes_A d) = c_1\sigma(c_2 \otimes_A d) = \sigma(c \otimes_A d_1)d_2$. Conversely, given a $(\mathcal{C}, \mathcal{C})$-bicomodule map $m : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$, let $\sigma = \epsilon_{\mathcal{C}} \circ m$. The verification that $m_\sigma$ is a well-defined $(\mathcal{C}, \mathcal{C})$-bicomodule map (note that by convention a $(\mathcal{C}, \mathcal{C})$-bicomodule map must be $(A, A)$-linear), that $\sigma = \epsilon_{\mathcal{C}} \circ m \in \mathrm{bbf}(\mathcal{C})$ and that the correspondence is bijective is straightforward. $\square$

**Remarks 3.4.** *Let $\mathcal{C}$ be an $A$-coring.*

(i) *For $\sigma \in \mathrm{bbf}(\mathcal{C})$, the map $m = m_\sigma$ in Lemma 3.3 (iii) is associative since for $c, d, e \in \mathcal{C}$,*

$$\begin{aligned} m(c \otimes_A m(d \otimes_A e)) &= m(c \otimes_A d_1\sigma(d_2 \otimes_A e)) = m(c \otimes_A d_1)\sigma(d_2 \otimes_A e) \\ &= \sigma(c \otimes_A d_1)d_2\sigma(d_3 \otimes_A e) = \sigma(c \otimes_A d_1)m(d_2 \otimes_A e) \\ &= m(\sigma(c \otimes_A d_1)d_2 \otimes_A e) = m(m(c \otimes_A d) \otimes_A e). \end{aligned}$$

*Then $(\mathcal{C}, m_\sigma)$ is an associative ring, in general without a unit.*

(ii) *Furthermore, for $\sigma \in \mathrm{bbf}(\mathcal{C})$ and $\sigma^r$ as above, we have that $\sigma^r$ is a ring homomorphism from $(\mathcal{C}, m_\sigma)$ to $\mathcal{C}^{*op} = {}^*(\mathcal{C}^o)$. To see this, we compute for all*

*$c, d, e \in \mathcal{C}$,*

$$\begin{aligned}\sigma^r(m_\sigma(c \otimes_A d))(e) &= \sigma^r(\sigma(c \otimes_A d_1)d_2)(e) = \sigma(c \otimes_A d_1)\sigma(d_2 \otimes_A e) \\ &= \sigma(c \otimes_A d_1\sigma(d_2 \otimes_A e)) = \sigma(c \otimes_A \sigma(d \otimes_A e_1)e_2) \\ &= \sigma^r(c)(\sigma(d \otimes_A e_1)e_2) = \sigma^r(c)(\sigma^r(d)(e_1)e_2) \\ &= \sigma^r(d) *^r \sigma^r(c)(e).\end{aligned}$$

*Similarly, $\sigma^l$ is a ring homomorphism from $(\mathcal{C}, m_\sigma)$ to $({}^*\mathcal{C})^o$.*

(iii) *If $M$ is a right $\mathcal{C}$-comodule (i.e., $M \in \mathcal{M}_A$ with a coaction $\rho^M : M \to M \otimes_A \mathcal{C}$ which is a coassociative right $A$-module map), then $M$ is a right $(\mathcal{C}, m_\sigma)$ module via $m \cdot c = m_0\sigma(m_1 \otimes_A c)$. The computation to show associativity is in* [2, 26.7] *or is a straightforward exercise. Unless $\sigma \circ \Delta_{\mathcal{C}} = \epsilon_{\mathcal{C}}$, we need not have that $M \otimes_{(\mathcal{C}, m_\sigma)} \mathcal{C} \cong M$.*

Finally, we note that the notion of balanced bilinear forms for corings is integral to the definition of coseparable corings and co-Frobenius corings.

Coseparable corings form an important class of corings for which forgetful functors are separable functors. Recall [2, Section 26] that an $A$-coring $\mathcal{C}$ is called coseparable if there exists a $(\mathcal{C}, \mathcal{C})$-bicomodule map $\pi : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$ such that $\pi \circ \Delta_{\mathcal{C}} = \mathcal{C}$. By Lemma 3.3 (iii) or as noted in [2, 26.1(b)], such a map $\pi$ exists if and only if there exists $\sigma \in \mathrm{bbf}(\mathcal{C})$ (called a cointegral in [2]) such that $\sigma \circ \Delta_{\mathcal{C}} = \epsilon_{\mathcal{C}}$, i.e., $(\mathcal{C}, m_\sigma)$ is a nonunital ring whose product has a section. In other words, $(\mathcal{C}, m_\sigma)$ is a separable $A$-ring in the sense of [1] or [2, Section 26].

In [2, 27.15], a left (right) co-Frobenius coring is an $A$-coring $\mathcal{C}$ such that there is an injective morphism from $\mathcal{C}$ to ${}^*\mathcal{C}$ ($\mathcal{C}^*$ respectively). Thus, by Lemma 3.3 (iii), $\mathcal{C}$ is left (right) co-Frobenius via an injective morphism which is an $(A, A)$-bimodule map if and only if there is a left (right) nondegenerate $\sigma \in \mathrm{bbf}(\mathcal{C})$. Note that this latter is precisely the definition of co-Frobenius in [7, Definition 2.4] (i.e. the $A$-bilinearity is specified) and that there the opposite multiplication to that of [2] is used on the left and right duals so that $A$ (not $A^o$) embeds in ${}^*\mathcal{C}$ and $\mathcal{C}^*$.

Now, a consistent definition of symmetric coring would require that the coring is co-Frobenius on the left and on the right with some compatibility conditions between the two structures.

## 4. The notion of a symmetric coring

In this section, we explore whether the notion of a symmetric coring is a sensible one, and suggest one possible definition which takes into account the fact that $A$ is not necessarily commutative.

The idea of a symmetric coalgebra $C$ over a field $k$ was given in [4] where the authors proved the following.

**Theorem 4.1.** *Let $C$ be a $k$-coalgebra, $k$ a field. Then the following are equivalent:*

(i) *There exists an injective morphism $\alpha : C \to C^*$ of $(C^*, C^*)$-bimodules.*

(ii) *There exists a bilinear form $B : C \times C \to k$ which is symmetric, nondegenerate and $C^*$-balanced.*

A symmetric coalgebra over a field is then defined to be one satisfying the equivalent conditions of Theorem 4.1.

One might expect that a suitable definition for a symmetric coring would be that an $A$-coring $\mathcal{C}$ is symmetric if there exists a right and left nondegnerate symmetric

$\sigma \in \mathrm{bbf}(\mathcal{C})$. We next show the equivalence of four conditions which mimic those of Theorem 4.1 in case $\mathcal{C}$ is right and left locally projective over $A$.

**Lemma 4.2.** *Let $\mathcal{C}$ be an $A$-coring. Let $\sigma \in \mathrm{bbf}(\mathcal{C})$ and consider the $k$-module map $\sigma^{op}$ from $\mathcal{C}\otimes_k\mathcal{C}$ to $A$ defined to be the composite of the twist map $\tau : \mathcal{C}\otimes_k\mathcal{C} \to \mathcal{C}\otimes_k\mathcal{C}$ and the surjection from $\mathcal{C}\otimes_k\mathcal{C}$ to $\mathcal{C}\otimes_A\mathcal{C}$ followed by $\sigma$. Then $\sigma^{op} \in \mathrm{bbf}(\mathcal{C})$ if and only if the following three conditions hold.*

(i) $\mathrm{Im}(\sigma) \subseteq Z(A)$.

(ii) $\sigma(ca \otimes_A d) = a\sigma(c \otimes_A d)$*, for all* $a \in A, c, d \in \mathcal{C}$.

(iii) $\sigma(c_1 \otimes_A d)c_2 = d_1\sigma(c \otimes_A d_2)$*, for all* $c, d \in \mathcal{C}$.

*Proof.* If $\sigma^{op} \in \mathrm{bbf}(\mathcal{C})$, then $\sigma^{op}$ must be well defined on $\mathcal{C}\otimes_A\mathcal{C}$. Thus we must have $\sigma^{op}(ca \otimes_A d) = \sigma^{op}(c \otimes_A ad)$, i.e., $\sigma(d \otimes_A ca) = \sigma(ad \otimes_A c)$. This holds if and only if $\sigma(d \otimes_A c)a = a\sigma(d \otimes_A c)$, so that $\mathrm{Im}(\sigma) \subseteq Z(A)$. Also, $\sigma^{op}$ must be an $(A, A)$-bimodule map, so that $\sigma(c \otimes_A ad) = \sigma(c \otimes_A d)a$ or, equivalently, using the facts that $\sigma \in \mathrm{bbf}(\mathcal{C})$ and $\mathrm{Im}(\sigma) \subseteq Z(A)$, $\sigma(ca \otimes_A d) = a\sigma(c \otimes_A d)$. The last condition is equivalent to the fact that $\sigma^{op}$ must satisfy the balanced property (3.1) of Definition 3.1. □

If $A = k$, then (i) and (ii) in the lemma above are automatic and (iii) holds if and only if $\sigma \in \mathrm{bbf}(\mathcal{C}^{\mathrm{o}})$.

Next we fix some notation.

Let $\Gamma$ denote the subring of ${}^*\mathcal{C}^*$ of maps from $\mathcal{C}$ to $A$ with image in $Z(A)$, the centre of $A$. On $\Gamma$ the left (right) $A$-actions induced from those on ${}^*\mathcal{C}$ and $\mathcal{C}^*$ coincide. Thus we may define an $(A, A)$-bimodule structure on $\Gamma$ by $(ac^*b)(c) = c^*(bca) = bc^*(c)a$. Note that if $f \in \Gamma$, then $f(c)(ab - ba) = 0$, i.e., the image of $f$ annihilates $[A, A]$, the additive commutator of $A$.

**Proposition 4.3.** *Let $\mathcal{C}$ be an $A$-coring. If $\mathcal{C}$ is left and right locally projective over $A$, then the following conditions are equivalent:*

(i) *There exists an injective morphism $\alpha$ of $(A, A)$-bimodules from $\mathcal{C}$ to the subring $\Gamma$ of ${}^*\mathcal{C}^*$ defined above such that $Im(\alpha)$ is a $({}^*\mathcal{C}, \mathcal{C}^*)$-bimodule with left and right actions given by $*^l$ and $*^r$, and $\alpha$ is a $({}^*\mathcal{C}, \mathcal{C}^*)$-bimodule map.*

(ii) *There exists an injective map $\alpha : \mathcal{C} \to {}^*\mathcal{C}^*$ such that $\alpha_1 := i_1 \circ \alpha$ is an $(A, A)$-bimodule, right $\mathcal{C}^*$-module morphism from $\mathcal{C}$ to $\mathcal{C}^*$, and $\alpha_2 := i_2 \circ \alpha$ is an $(A, A)$-bimodule, left ${}^*\mathcal{C}$-module morphism from $\mathcal{C}$ to ${}^*\mathcal{C}$ where $i_1 : {}^*\mathcal{C}^* \hookrightarrow \mathcal{C}^*$ and $i_2 : {}^*\mathcal{C}^* \hookrightarrow {}^*\mathcal{C}$ are the inclusion maps.*

(iii) *There exists a right nondegenerate $\sigma \in \mathrm{bbf}(\mathcal{C})$ such that $\sigma^{op} \in \mathrm{bbf}(\mathcal{C})$ also.*

(iv) *There exists a right and left nondegenerate $\sigma \in \mathrm{bbf}(\mathcal{C})$ such that $\sigma = \sigma^{op}$.*

*Proof.* (i) implies (ii). It is straightforward to check that if a map $\alpha$ satisfies (i), then it also satisfies (ii).

(ii) implies (iii). Suppose that (ii) holds. Then by Lemma 3.3, $\alpha_1 = \sigma^r$ for some right nondegenerate $\sigma \in \mathrm{bbf}(\mathcal{C})$ and $\alpha_2 = \omega^l$ for some left nondegenerate $\omega \in \mathrm{bbf}(\mathcal{C})$, i.e., $\alpha(c)(d) = \sigma(c \otimes_A d) = \omega(d \otimes_A c)$. Then $\omega = \sigma^{op}$ and $\sigma^{op} \in \mathrm{bbf}(\mathcal{C})$.

(iii) implies (i). Suppose $\sigma \in \mathrm{bbf}(\mathcal{C})$ is right nondegenerate and $\sigma^{op} \in \mathrm{bbf}(\mathcal{C})$. Then $\sigma^r$ is an $(A, A)$-bimodule map from $\mathcal{C}$ to $\mathcal{C}^*$, and $(\sigma^{op})^l$ is an $(A, A)$-bimodule map from $\mathcal{C}$ to ${}^*\mathcal{C}$. Since $\sigma^r(c)(d) = \sigma(c \otimes_A d) = \sigma^{op}(d \otimes_A c) = (\sigma^{op})^l(c)(d)$, then $\sigma^r = (\sigma^{op})^l$. Define $\alpha = \sigma^r = (\sigma^{op})^l$. Then $\alpha$ maps $\mathcal{C}$ to ${}^*\mathcal{C}^*$ and since $\sigma$ is right nondegenerate, $\alpha$ is injective. Also $Im(\alpha)$ is a right $\mathcal{C}^*$-module and $\alpha$ is a right $\mathcal{C}^*$-module map by Lemma 3.3 since $\sigma^r(c) *^r c^* = \sigma^r(c \leftharpoonup c^*) \in Im(\alpha)$.

Similarly, $Im(\alpha)$ is a left ${}^*\mathcal{C}$-module and $\alpha$ is a left ${}^*\mathcal{C}$-module map. By Lemma 4.2, $Im(\sigma) \subseteq Z(A)$ so that $Im(\alpha) \subseteq \Gamma$. Finally, since for $f \in {}^*\mathcal{C}, g \in \mathcal{C}^*$, $c \in \mathcal{C}$, we have $(f*^l\alpha(c))*^r g = \alpha(f \rightharpoonup c)*^r g = \alpha((f \rightharpoonup c) \leftharpoonup g) = \alpha((f \rightharpoonup (c \leftharpoonup g)) = f*^l(\alpha(c)*^r g)$ so that $\alpha$ is a $({}^*\mathcal{C}, \mathcal{C}^*)$-bimodule map.

Thus we have shown the equivalence of (i),(ii) and (iii). Clearly (iv) implies (iii) and it remains to show that the equivalent conditions (i), (ii) and (iii) imply (iv).

Let $\alpha$ satisfy (i), (ii). By Remark 3.4, we have that $\alpha = \sigma^r$ is a multiplication preserving isomorphism from $(\mathcal{C}, m_\sigma)$ to $Im(\alpha)$ and $\alpha = (\sigma^{op})^l$ is also a multiplication preserving isomorphism from $(\mathcal{C}, m_{\sigma^{op}})$ to $Im(\alpha)$. Thus $(\sigma^r)^{-1} \circ (\sigma^{op})^l = \alpha^{-1} \circ \alpha = Id_{\mathcal{C}}$ is a multiplication preserving bijection from $(\mathcal{C}, m_{\sigma^{op}})$ to $(\mathcal{C}, m_\sigma)$. Then $m_\sigma(c \otimes_A d) = m_{\sigma^{op}}(c \otimes_A d)$, i.e., $c_1\sigma(c_2 \otimes_A d) = c_1\sigma(d \otimes_A c_2)$ so that, applying $\epsilon_{\mathcal{C}}$, we see that $\sigma = \sigma^{op}$. □

Note that Proposition 4.3 provides a new proof of the equivalence of the statements in Theorem 4.1 which does not depend on the fact that the rational dual of a co-Frobenius coalgebra over a field has local units.

A definition of symmetric coring parallel to the definition of symmetric coalgebra over a field would be to require that the coring satisfy condition (iv) in Proposition 4.3. However, this seems very restrictive, depending on commutativity of elements in $A$. Instead, we suggest the following.

**Definition 4.4.** *Let $A$ be a ring, not necessarily commutative, and let $A'$ be the ideal of $A$ generated by the additive commutator $[A, A] = \{ab - ba \mid a, b \in A\}$. Let $\mathcal{C}$ be an $A$-coring. We say that $\mathcal{C}$ is symmetric if there exists $\sigma \in \mathrm{bbf}(\mathcal{C})$ such that $\sigma$ is left and right nondegenerate and for all $c, d$ in $\mathcal{C}$, we have that $\sigma(c \otimes_A d) - \sigma(d \otimes_A c) \in A'$.*

**Examples 4.5.** (i) *The trivial coring and the matrix coring from Examples 2.1, with the nondegenerate balanced forms defined in Examples 3.2 are both symmetric in the sense of Definition 4.4 but do not satisfy the equivalent conditions of Proposition 4.3, unless $A$ is commutative.*

(ii) *The coring $\mathcal{C} = A \otimes_k H$ from Examples 2.1 with nontrivial $H$-coaction on $A$ and with $\sigma$ as in Examples 3.2 is not symmetric in either sense even if $t(hS(g)) = t(gS(h))$ for all $g, h \in H$. (Of course, if the coaction is trivial, then the coring is clearly symmetric in the sense of Definition 4.4 if $B(h, g) = t(hS(g))$ is a symmetric form for $H$.)*

Corings satisfying the equivalent conditions of Proposition 4.3 clearly are symmetric in the sense of Definition 4.4.

The next example builds a symmetric coring from a symmetric coseparable $k$-coalgebra, $k$ a field.

**Example 4.6.** *Let $k$ be a field and let $C$ be a $k$-coalgebra with a $C^*$-balanced nondegenerate symmetric bilinear form $B : C \otimes_k C \to k$ such that $B \circ \Delta_C = \epsilon_C$; in other words, $C$ is a symmetric coalgebra via $B$ and also a coseparable coalgebra via $B$. For example, let $C$ be a cosemisimple involutory $k$-Hopf algebra $H$ with antipode $S$. Then if $\lambda$ is a left and right integral for $H$ in $H^*$ such that $\lambda(1) = 1$, then $B :\ H \otimes_k H \to k$ defined by $B(h, g) = \lambda(hS(g))$ satisfies the conditions above.*

*Let $A = C^*$, the dual algebra of $C$ with convolution multiplication $*$ and let $\mathcal{C} = C$; we show first that $\mathcal{C}$ has the structure of an $A$-coring.*

*$\mathcal{C}$ is an $(A, A)$-bimodule with the standard $C^*$-bimodule structure on $C$, namely $c^* \rightharpoonup c \leftharpoonup d^* = d^*(c_1)c_2c^*(c_3)$, for all $c \in C$, $c^*, d^* \in C^*$. Define the coproduct*

*on $\mathcal{C}$ to be the composite of the co-opposite comultiplication of $C$ and the surjection from $C \otimes_k C$ to $C \otimes_A C$, namely $\Delta_{\mathcal{C}}(c) = c_2 \otimes_A c_1$. Since $\Delta_{\mathcal{C}}(d^*(c_1)c_2c^*(c_3)) = c^*(c_4)c_3 \otimes_A c_2 d^*(c_1) = c^* \rightharpoonup c_2 \otimes_A c_1 \leftharpoonup d^*$, then $\Delta_{\mathcal{C}}$ is an $(A, A)$-bimodule map and the coassociativity follows from that of $\Delta_C$.*

*Now, we define the counit map for $\mathcal{C}$. For $B$ the balanced bilinear form for the $k$-coalgebra $C$ as above, define $\epsilon_{\mathcal{C}} : C \to C^*$ by $\epsilon_{\mathcal{C}}(c) = B(c, -) = B(-, c)$. Note that $\epsilon_{\mathcal{C}}$ is injective since $B$ is left and right nondegenerate. Then*

$$\epsilon_{\mathcal{C}}(d^* \rightharpoonup c)(x) = B(x, c_1)d^*(c_2) = d^*(x_1)B(x_2, c) = (d^* * \epsilon_{\mathcal{C}})(x),$$

*so $\epsilon_{\mathcal{C}}$ is a left $A$-module map and, similarly, $\epsilon_{\mathcal{C}}$ is a right $A$-module map.*

*The compatibility of $\Delta_{\mathcal{C}}$ and $\epsilon_{\mathcal{C}}$ follows from the coseparability of $C$ by*

$$(\epsilon_{\mathcal{C}} \otimes_A \mathcal{C})(c_2 \otimes_A c_1) = \epsilon_{\mathcal{C}}(c_2) \rightharpoonup c_1 = c_1 B(c_2, c_3) = c$$

*and, similarly, $(\mathcal{C} \otimes_A \epsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = \mathcal{C}$. Thus we have shown that $\mathcal{C}$ is an $A$-coring.*

*Now, define $\mathcal{B} : \mathcal{C} \otimes_A \mathcal{C} \to A$ by $\mathcal{B}(c \otimes_A d) = \epsilon_{\mathcal{C}}(c) * \epsilon_{\mathcal{C}}(d)$. Since $\epsilon_{\mathcal{C}}$ is an $(A, A)$-bimodule map, then $\mathcal{B}$ is also. We show that $\mathcal{B}$ is a well-defined balanced form. To see that $\mathcal{B}$ is well defined, we compute for $c, d, x \in \mathcal{C}$, $c^* \in A$,*

$$\begin{aligned}
\mathcal{B}(c \leftharpoonup c^* \otimes_A d)(x) &= c^*(c_1)\mathcal{B}(c_2 \otimes_A d)(x) \\
&= c^*(c_1)B(c_2, x_1)B(x_2, d) \\
&= B(c, x_1)c^*(x_2)B(x_3, d) \\
&= B(c, x_1)B(x_2, d_1)c^*(d_2) \\
&= \mathcal{B}(c \otimes_A c^* \rightharpoonup d)(x).
\end{aligned}$$

*To see that $\mathcal{B}$ is balanced, we must show that $c_2 \leftharpoonup \mathcal{B}(c_1 \otimes_A d) = \mathcal{B}(c \otimes_A d_2) \rightharpoonup d_1$, for all $c, d \in \mathcal{C}$. We already showed that $c_2 \leftharpoonup \epsilon_{\mathcal{C}}(c_1) = c$ and thus we have that*

$$c_2 \leftharpoonup \mathcal{B}(c_1 \otimes_A d) = c_2 \leftharpoonup \epsilon_{\mathcal{C}}(c_1) * \epsilon_{\mathcal{C}}(d) = c \leftharpoonup \epsilon_{\mathcal{C}}(d) = B(d, c_1)c_2.$$

*Similarly, $\mathcal{B}(c \otimes_A d_2) \rightharpoonup d_1 = d_1 B(c, d_2)$ and these are equal since $B$ is balanced and symmetric.*

*We now show that $\mathcal{B}$ is right nondegenerate, i.e., that $\mathcal{B}(d \otimes_A -) = 0$ implies $d = 0$. Since $\mathcal{B}(d \otimes_A c) = B(d, -) * B(-, c)$, if $\mathcal{B}(d \otimes_A -) = 0$, then $B(d, x_1)B(c, x_2) = 0$ for all $c, x \in C$, and, in particular, $0 = B(d, c_1)B(c_3, c_2) = B(d, c)$, for all $c \in C$, contradicting the fact that $B$ is right nondegenerate. Similarly, $\mathcal{B}$ is left nondegenerate.*

*Clearly, $\mathcal{B}$ is symmetric in the sense of Definition 4.4 but unless $B(c, -) * B(d, -) = B(d, -) * B(c, -)$ in $A = C^*$, for all $c, d \in C$, the equivalent conditions of Proposition 4.3 do not hold.*

Finally, we show that if $\mathcal{C}$ is an $A$-coring which is symmetric in the sense of Definition 4.4 via a map $\sigma \in \mathrm{bbf}(\mathcal{C})$ satisfying a further nondegeneracy condition, then the coring induced by the surjection from $A$ to $A/A'$ satisfies Proposition 4.3 (iv).

**Theorem 4.7.** *Let $\mathcal{C}$ be an $A$-coring and let $\sigma \in \mathrm{bbf}(\mathcal{C})$ such that*

(i) *$\sigma$ is left and right nondegenerate.*

(ii) *For all $c, d \in \mathcal{C}$, we have that $\sigma(c \otimes_A d) - \sigma(d \otimes_A c) \in A'$.*

(iii) *$\sigma(c \otimes_A d) \in A'$, for all $d \in \mathcal{C}$, implies that $c \in A'\mathcal{C} + \mathcal{C}A'$.*

*Then the surjection from $A$ to $B = A/A'$ induces a $B$-coring structure on $\mathcal{D} := B \otimes_A \mathcal{C} \otimes_A B$ such that $\mathcal{D}$ satisfies condition (iv) of Proposition 4.3.*

*Proof.* Let $\overline{a}$ denote the image of $a \in A$ in $B = A/A'$; for $\overline{1}$, we may write $1_B$. We note first that $\mathcal{D} = B \otimes_A \mathcal{C} \otimes_A B = 1_B \otimes_A \mathcal{C} \otimes_A 1_B$ since for $a \in A$, $\overline{a} = (1_B)a = a(1_B)$. In particular, if $c \in A'\mathcal{C} + \mathcal{C}A'$, then $1_B \otimes_A c \otimes_A 1_B = 0$ in $\mathcal{D}$. Recall [2, 17.2] that $\mathcal{D}$ is a $B$-coring with counit $\epsilon_{\mathcal{D}}$ and comultiplication $\Delta_{\mathcal{D}}$ defined by

$$\epsilon_{\mathcal{D}}(1_B \otimes_A c \otimes_A 1_B) = \overline{\epsilon_{\mathcal{C}}(c)}, \text{ and}$$
$$\Delta_{\mathcal{D}}(1_B \otimes_A c \otimes_A 1_B) = (1_B \otimes_A c_1 \otimes_A 1_B) \otimes_B (1_B \otimes_A c_2 \otimes_A 1_B).$$

Note that if $a' \in A'$ then for $c, d \in \mathcal{C}$ we have $\sigma(c \otimes_A a'd) = \sigma(a'd \otimes_A c) + a''$, for some $a'' \in A'$, and so $\sigma(c \otimes_A a'd) = a'\sigma(d \otimes_A c) + a'' \in A'$ and

$$\overline{\sigma(c \otimes_A a'd)} = \overline{\sigma(ca' \otimes_A d)} = \overline{0}.$$

Thus the $(B, B)$-bimodule map $\mathcal{B} : \mathcal{D} \otimes_B \mathcal{D} \to B$ given by

$$\mathcal{B}((1_B \otimes_A c \otimes_A 1_B) \otimes_B (1_B \otimes_A d \otimes_A 1_B)) = \overline{\sigma(c \otimes_A d)},$$

is well defined. For if $x' - x \in A'$, then

$$\sigma(cx' \otimes_A d) - \sigma(cx \otimes_A d) = \sigma(c(x' - x) \otimes_A d) \in A'.$$

Furthermore, $\mathcal{B}$ is balanced. To see this, we compute for $c, d \in \mathcal{C}$,

$$\begin{aligned}
&\mathcal{B}((1_B \otimes_A c \otimes_A 1_B) \otimes_B (1_B \otimes_A d_1 \otimes_A 1_B))(1_B \otimes_A d_2 \otimes_A 1_B) \\
&= \overline{\sigma(c \otimes_A d_1)} \otimes_A d_2 \otimes_A 1_B \\
&= 1_B \otimes_A \sigma(c \otimes_A d_1)d_2 \otimes_A 1_B \\
&= 1_B \otimes_A c_1\sigma(c_2 \otimes_A d) \otimes_A 1_B \\
&= 1_B \otimes_A c_1 \otimes_A \overline{\sigma(c_2 \otimes_A d)} \\
&= (1_B \otimes_A c_1 \otimes_A 1_B)\mathcal{B}((1 \otimes_A c_2 \otimes_A 1) \otimes_B (1_B \otimes_A d \otimes_A 1_B)).
\end{aligned}$$

Thus, we have that $\mathcal{B} \in \mathrm{bbf}(\mathcal{D})$. We show that $\mathcal{B}$ is left and right nondegenerate and that $\mathcal{B} = \mathcal{B}^{op}$.

Suppose that $\mathcal{B}((1_B \otimes_A c \otimes_A 1_B) \otimes_B -)$ maps all elements of $\mathcal{D}$ to $\overline{0} \in B$. Then $\sigma(c \otimes_A d) \in A'$, for all $d \in \mathcal{C}$. By (iii) in the statement of the theorem, we must have that $c \in A'\mathcal{C} + \mathcal{C}A'$. But then $1_B \otimes_A c \otimes_A 1_B = 0 \in \mathcal{D}$, and so $\mathcal{B}$ is right nondegenerate. Similarly, $\mathcal{B}$ is left nondegenerate.

Finally, we note that $\mathcal{B} = \mathcal{B}^{op}$ is straightforward, so the proof is complete. □

**Example 4.8.** *We noted in Example 4.5 that the trivial coring and matrix coring do not necessarily satisfy the conditions in Proposition 4.3. However, in both cases, Theorem 4.7 applies.*

**Further questions:** (i) Another equivalent condition for a coalgebra $C$ over a field $k$ to be symmetric involves the multiplication on $C$ induced by the multiplication on the ring with local units $C^{*\mathrm{rat}}$ [4, Theorem 3.3 (3)]. This parallels an equivalent condition for a finite dimensional algebra over a field to be symmetric [4, Theorem 3.1 (3)], namely that a $k$-algebra $A$ is symmetric if there exists a $k$-linear map $f : A \to k$ such that $f(xy) = f(yx)$ for $x, y \in A$, and $\mathrm{Ker}(f)$ does not contain a non-zero left ideal. An analogous condition should hold in the coring case.

(ii) In [4, Section 5], a functorial characterization of symmetric coalgebras over a field is given. It is unclear what the corresponding results (if any) for symmetric corings should be.

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## References

[1] T. Brzeziński, L. Kadison and R. Wisbauer, On coseparable and biseparable corings, In: Hopf algebras in non-commutative geometry and physics, S. Caenepeel, F. Van Oystaeyen (eds.), Lecture Notes in Pure and Applied Mathematics 239, Marcel Dekker, New York, 2004, p. 71-88 (arXiv:math.RA/0208122).
[2] T. Brzeziński and R. Wisbauer, Corings and Comodules, London Math. Soc. Lecture Note Series 309, Cambridge University Press, 2003.
[3] S. Caenepeel, G. Militaru and S. Zhu, Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations, LNM 1787, Springer Berlin-Heidelberg, 2002.
[4] F. Castaño Iglesias, S. Dăscălescu and C. Năstăsescu, Symmetric coalgebras, J. Algebra 279 (2004), 326-344.
[5] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, Hopf algebras: an introduction, Marcel Dekker, Pure and Applied Mathematics 235, 2001.
[6] J. Gómez-Torrecillas and A. Louly, Coseparable corings, Communications in Algebra 31 (2003), 4455-4471.
[7] M. Iovanov and J. Vercruysse, CoFrobenius corings and related functors, preprint. (arXiv:math.RA/0610853)
[8] B. I. Lin, Semiperfect coalgebras, J. Algebra 49 (1977), 357-373.
[9] M.E.Sweedler, The predual theorem to the Jacobson-Bourbaki theorem, Transactions of the Amer. Math. Soc. 213 (1975), 391-406.

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