THE BIJECTIVITY OF THE ANTIPODE REVISITED

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Dedicated to Mia Cohen on the occasion of her retirement

ABSTRACT. We provide a very short approach to several fundamental results for Hopf algebras with nonzero integrals. Besides being short, our approach is the first to prove the bijectivity of the antipode without using the uniqueness of the integrals of Hopf algebras and to obtain the uniqueness of integrals as a corollary in a way similar to the classical theory of the Haar measure on compact groups.

INTRODUCTION

One of the fundamental notions of the theory of Hopf algebras is that of an integral, which is an analog of the Haar integral of a compact group and draws its name from there. More precisely, if G is a compact group and $R(G)$ is the algebra of continuous representative functions on G, i.e. the space spanned by the coefficients η_{ij} of all continuous representations $\eta : G \to GL_n(\mathbb{C})$, then the restriction of the Haar integral to $R(G)$ becomes an integral in the Hopf algebra sense (see Abe (1977) or Dăscălescu et al. (2001)). In this respect, a Hopf algebra having a nonzero integral is a generalization of the algebra of continuous representative functions on a compact group. Integrals for Hopf algebras were introduced by Sweedler (1969a). In that paper he proves a series of fundamental results about Hopf algebras with nonzero integrals, including the fact that integrals are unique and the antipode is bijective when the Hopf algebra is finite dimensional. The questions about the validity of these results for Hopf algebras with nonzero integrals of possibly infinite dimension appear explicitly in Sweedler (1969b). These questions were given affirmative answers: the uniqueness of the integrals was proved by Sullivan (1971), then, using Sullivan's result, Radford (1977) proved that the antipode of a Hopf algebra with nonzero integrals is bijective. Many other proofs for the uniqueness of integrals were found later. Some of these proofs have a strong homological flavor and use the fact that integrals are just comodule maps: Ştefan (1995), Beattie et al. (1998), Menini et al. (2001), Dăscălescu et al. (1999). In contrast with the abundance of proofs for the uniqueness of integrals, Radford's proof for the bijectivity of the antipode was virtually the only one available (with a simplification due to Călinescu (2001)) until very recently, when alternate proofs were obtained by Iovanov (preprint1) and (preprint2) by using a purely coalgebraic approach, as a byproduct of a general theory of algebraic "integrals" or infinite dimensional generalized Frobenius algebras. All proofs for the bijectivity of the antipode used the uniqueness of integrals, and it was hard to say whether this happened by necessity or it was just an effect of the order in which the two results were obtained. Moreover, the

classical proof for the uniqueness of Haar measures adapted for Hopf algebras requires the bijectivity of the antipode (see Van Daele (1998) and Raianu (2000)). In the classical case of compact groups, the Hopf algebra of representative continuous functions clearly has a bijective antipode because it is commutative, and this probably made the causative relationship between bijectivity and uniqueness harder to understand. The fact that the antipode of a Hopf algebra with nonzero integrals might not be necessarily bijective was the only obstacle in proving the uniqueness of the integrals by using the same technique as in the case of Haar measures.

In this note we find a very short approach to explain the above mentioned results. We first prove the bijectivity of the antipode without using the uniqueness of the integrals. This is the first proof constructed in this manner, and it follows by using a technique from Iovanov (preprint1). We can then just use the classical proof of the uniqueness of the Haar measure from locally compact groups, as was done in Van Daele (1998) for multiplier Hopf algebras (see also the chapter on Haar measures in Bourbaki (1963)). Thus, besides being short, this proof also has the advantage that it shows once more an even stronger parallel than noted previously between Hopf algebras and locally compact groups.

The Proofs

Let H be a Hopf algebra over the field k. Recall that a left integral λ of the Hopf algebra H is an element in H^* such that $\alpha\lambda = \alpha(1)\lambda$ for all $\alpha \in H^*$. We also recall that whenever nonzero left integrals exist, Sweedler (1969b) proved that the antipode S of the Hopf algebra H is injective, and therefore it has a left inverse S^l. Sweedler proved the injectivity of the antipode after twisting by S the module structure in a Hopf module structure on the rational module $Rat(H^*)$. Therefore, it makes sense that when trying to prove the surjectivity one should consider twisting by S the comodule structure in some natural Hopf module structure on the rational part. This is precisely what we are going to do.

For $(M, \rho) \in \mathcal{M}^H$,
$$\rho : M \longrightarrow M \otimes H, \quad \rho(m) = m_0 \otimes m_1,$$
we define $^S M \in {}^H\mathcal{M}$ with comodule structure given by
$$m \longmapsto m_{(-1)} \otimes m_{(0)} = S(m_1) \otimes m_0$$
It is clear that we have a functor $F : \mathcal{M}^H \longrightarrow {}^H\mathcal{M}$, $F(M) = {}^S M$, and F is the identity on morphisms.

If $x, y \in H$ and $\alpha \in H^*$, we denote $(x \rightharpoonup \alpha)(y) = \alpha(yx)$ and $(\alpha \leftharpoonup x)(y) = \alpha(xy)$. Then we have:

Proposition 1. $^S Rat(H^*)$, *with left H-module structure given by*
$$H \otimes {}^S Rat(H^*) \longrightarrow {}^S Rat(H^*), \quad x \otimes \alpha \longmapsto x \rightharpoonup \alpha, \quad x \in H, \ \alpha \in Rat(H^*)$$
and left H-comodule structure as above is a left H-Hopf module.

Proof. The first problem is that it is not obvious why $^S Rat(H^*)$ is a left H-module under the $\rightharpoonup$ action. To see this, let $\alpha \in Rat(H^*)$, which means that there exist $x_i^\alpha \in H$ and $g_i^\alpha \in H^*$ such that for all $\beta \in H^*$ and $h \in H$ we have

(1) $$\beta\alpha(h) = \beta(h_1)\alpha(h_2) = \beta(x_i^\alpha)g_i^\alpha(h)$$

Now let $x \in H$, denote as before the left inverse of S by S^l, and let us compute

$$
\begin{aligned}
\beta(x\rightharpoonup\alpha)(h) &= \beta(h_1)(x\rightharpoonup\alpha)(h_2) \\
&= \beta S^l(S(h_1))\alpha(h_2 x) \\
&= \beta S^l(x_1 S(x_2)S(h_1))\alpha(h_2 x_3) \\
&= (\beta S^l \leftharpoonup x_1)(S((hx_2)_1))\alpha((hx_2)_2) \\
&= (\beta S^l \leftharpoonup x_1) \circ S(x_i^\alpha)g_i^\alpha(hx_2) \quad - \quad \text{by (1)} \\
&= \beta(S^l(x_1 S(x_i^\alpha)))(x_2 \rightharpoonup g_i^\alpha)(h)
\end{aligned}
$$

Therefore, we proved that $x\rightharpoonup\alpha \in Rat(H^*)$.

To finish the proof, we need to show that

$$
(x\rightharpoonup\alpha)_{(-1)} \otimes (x\rightharpoonup\alpha)_{(0)} = x_1\alpha_{(-1)} \otimes x_2\rightharpoonup\alpha_{(0)}
$$

which is

$$
S((x\rightharpoonup\alpha)_1) \otimes (x\rightharpoonup\alpha)_0 = x_1 S(\alpha_1) \otimes x_2\rightharpoonup\alpha_0
$$

or

$$
< \beta S((x\rightharpoonup\alpha)_1)(x\rightharpoonup\alpha)_0, y > = < \beta(x_1 S(\alpha_1))(x_2\rightharpoonup\alpha_0), y >, \quad \forall \beta \in H^*, \ y \in H
$$

We have

$$
\begin{aligned}
< \beta S((x\rightharpoonup\alpha)_1)(x\rightharpoonup\alpha)_0, y > &= < (\beta \circ S) * (x\rightharpoonup\alpha), y > \quad \text{(rt H-com str of $Rat(H^*)$)} \\
&= \beta S(y_1)(x\rightharpoonup\alpha)(y_2) \\
&= \beta S(y_1)\alpha(y_2 x) \\
&= \beta S(y_1)\alpha(y_2 x_2)\varepsilon(x_1) \\
&= \beta(\varepsilon(x_1)S(y_1))\alpha(y_2 x_2) \\
&= \beta(x_1 S(x_2)S(y_1)\alpha(y_2 x_3) \\
&= (\beta \leftharpoonup x_1)(S(y_1 x_2))\alpha(y_2 x_3) \\
&= < (\beta \leftharpoonup x_1) \circ S, (yx_2)_1 > < \alpha, (yx_2)_2 > \\
&= < ((\beta \leftharpoonup x_1) \circ S) * \alpha, yx_2 > \\
&= < ((\beta \leftharpoonup x_1) \circ S)(\alpha_1)\alpha_0, yx_2 > \\
&= \beta(x_1 S(\alpha_1))\alpha_0(yx_2) \\
&= < \beta(x_1 S(\alpha_1))(x_2\rightharpoonup\alpha_0), y >,
\end{aligned}
$$

which ends the proof. $\qquad\square$

Let C be a coalgebra and $M \in {}^C\mathcal{M}$. The coalgebra C_M associated to M is the smallest subcoalgebra C_M of C such that $\rho(M) \subseteq C_M \otimes M$, i.e. $C_M = \cap_{A \subseteq C, \rho(M) \subseteq A \otimes M} A$ (see Dăscălescu et al. (2001, p. 102)). With this notation we have:

Proposition 2. *If $M \xrightarrow{f} N$ is a surjective morphism of left C-comodules, then $C_N \subseteq C_M$.*

Proof. Let $K = Ker(f)$. Then clearly $C_K \subseteq C_M$. Therefore, K is a C_M-subcomodule, so $\rho_{M/K}(M/K) \subseteq C_M \otimes M/K$, i.e. $\rho_N(N) \subseteq C_M \otimes N$, hence $C_N \subseteq C_M$ by definition. $\qquad\square$

We are now ready to prove

Theorem 3. *If H is co-Frobenius, S is bijective.*

Proof. By Proposition 1 and the fundamental theorem of Hopf modules, we have that $^SRat(H^*) \simeq H \otimes (^SRat(H^*))^{co} = H \otimes \int_l$, since it is easy to see that $(^SRat(H^*))^{co} = \int_l$. Also, since $(Rat(H^*))^H \simeq \int_l \otimes H = H^{(dim \int_l)}$ in $\mathcal{M}^H$, we get $^SRat(H^*) \simeq (^SH)^{(dim \int_l)} = \bigoplus_{dim \int_l} {}^SH$, using the fact that the functor F clearly commutes with direct sums. Since $Rat(H^*) \neq 0$ (equivalently $\int_l \neq 0$) we can find a surjection of left H-comodules

$$\pi : (^SH)^{(dim \int_l)} \simeq {}^SRat(H^*) \simeq H \otimes (^SRat(H^*))^{co} \twoheadrightarrow {}^HH$$

Then $C_H \subseteq \sum C_{S_H} = C_{S_H}$ by Proposition 2 and the obvious fact that $C_{\bigoplus_{i \in I} M_i} = \sum_{i \in I} C_{M_i}$. Obviously, $C_H = H$ (by the counit property), and also $C_{S_H} = S(H)$, since $\forall h \in H, S(h) = S(h_2)\varepsilon(h_1) \in C_{S_H}$, because $\rho_{S_H}(h) = S(h_2) \otimes h_1 \in H \otimes {}^SH$. So $H \subseteq S(H)$, and the proof is complete. $\square$

Corollary 4. *If $t \in \int_l$, $t \neq 0$, then $t \circ S \in \int_r$, $t \circ S \neq 0$.*

Proof. Obvious. $\square$

As a consequence of this proof, the proof for the uniqueness of integrals can be translated verbatim to Hopf algebras from the case of Haar measures, as was done by Van Daele (1998) for regular multiplier Hopf algebras. This proof could not be used for Hopf algebras because it requires the bijectivity of the antipode, and until now all proofs of the bijectivity of the antipode used the uniqueness of integrals. A modified version of the proof below not requiring the bijectivity of the antipode was given by Raianu (2000).

Corollary 5. *The dimension of $\int_l$ is at most one.*

Proof. (Identical to the proof of Van Daele (1998, Theorem 3.7)) Let $t_1, t_2 \in \int_l$, $t_2 \neq 0$. By Corollary 4 $\lambda = t_2 \circ S \in \int_r \setminus \{0\}$. Then for any $h \in H$ there is a $g \in H$ such that

$$(2) \qquad t_1(xh) = t_2(xg) \quad \forall x \in H.$$

Indeed, let $l, m \in H$ such that $\lambda(l) = 1$ and $t_2(m) = 1$. Then

$$
\begin{aligned}
t_1(xh) &= \lambda(l)t_1(xh) \\
&= \lambda(x_1h_1l)t_1(x_2h_2) \quad (t_1 \in \int_l) \\
&= \lambda(x_1h_1l_1)t_1(x_2h_2l_2S(l_3)) \\
&= \lambda(xhl_1)t_1(S(l_2)) \quad (\lambda \in \int_r) \\
&= \lambda(xe) \quad (e = hl_1t_1(S(l_2))) \\
&= \lambda(xe)t_2(m) \\
&= \lambda(x_1e_1)t_2(x_2e_2m) \quad (\lambda \in \int_r) \\
&= \lambda(x_1e_1m_2S^{-1}(m_1))t_2(x_2e_2m_3) \\
&= \lambda(S^{-1}(m_1))t_2(xem_2) \quad (t_2 \in \int_l) \\
&= t_2(xg) \quad (g = em_2\lambda(S^{-1}(m_1)).
\end{aligned}
$$

We finish the proof by showing that t_1 is a scalar multiple of t_2. For $y \in H$ we have:

$$
\begin{aligned}
t_1(y) &= \lambda(l)t_1(y) \\
&= \lambda(l_1)t_1(yl_2) \quad (\lambda \in \smallint_r) \\
&= \lambda(S(y_1)y_2l_1)t_1(y_3l_2) \\
&= \lambda(S(y_1))t_1(y_2l) \quad (t_1 \in \smallint_l) \\
&= \lambda(S(y_1))t_2(y_2g) \quad - \quad \text{by } (2) \\
&= \lambda(g)t_2(y),
\end{aligned}
$$

where the last equality follows from reversing the previous three equalities, and the proof is complete. $\square$

Remarks 6. *a) Note that aside from the bijectivity of the antipode, the proof above uses only the definition of integrals.*
b) To compensate for not being able to use the inverse of S, a special left integral had to be chosen in Raianu (2000) and it was shown to form a basis of $\int_l$. Once we are able to use the inverse of S, the proof above shows that any non-zero left integral will do.

Following the work of Lin, Larson, Sweedler, and Sullivan, the existence of a nonzero integral is equivalent to various representation theoretical properties of the Hopf algebra, such as that of being co-Frobenius as a coalgebra, or having nonzero rational part. As a final application, we show how our approach may be used to simplify the proof of some of these results (see Dăscălescu et al. (2001, Theorem 5.3.2)):

Corollary 7. *For a Hopf algebra H the following assertions are equivalent:*
(1) H is left co-Frobenius
(2) H is left quasi-co-Frobenius
(3) H is left semiperfect
(4) $Rat(H^*) \neq 0$ (H^* as a left H^*-module)
(5) $\int_l \neq 0$
(6) The right hand version of (1)-(5)

Proof. All implications follow directly from the definitions or from Sweedler's isomorphism, with the exception of 5)⇒6) which follows from Corollary 4. $\square$

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