QUANTIFIER-ELIMINABLE LOCALLY FINITE GRAPHS

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Abstract. We identify the locally finite graphs that are quantifier-eliminable and their first order theories in the signature of distance predicates.

Our study of quantifier-eliminable locally finite graphs was motivated by the results in [12, 4]. The authors of [12] showed that the theory of the complete binary tree admits quantifier elimination (q.e.) in the signature L_∞ consisting of distance predicates. Their proofs are essentially syntactic. This prompted us to look for a proof closer in spirit to Robinson's message: to prove quantifier-elimination, use model theory rather than syntactic methods whenever possible[1]. We ended up answering the question of which locally finite graphs are quantifier-eliminable in the signature L_∞. They are precisely the 6-transitive graphs determined by Cameron in [2] and the infinite locally finite graphs with each connected component isomorphic to a member of a class of graphs that we call clique-trees. As a consequence of this result, we also show that complete rooted trees are quantifier-eliminable in $L_\infty \cup \{r\}$ where r is the constant symbol for root. This generalizes the corresponding result in [12] about the complete binary tree.

1. Finitely transitive Locally finite graphs

We summarize here the results of graph theory that we need. The reader can consult [7] for the graph theoretic terms that appear subsequently. Graphs in this article have neither loops nor multiple edges. A graph is *locally finite* if each of its vertex has finite valency. In particular, finite graphs are locally finite. We now give a family of infinite examples.

For natural numbers $m, n > 1$, consider the following four properties on a graph.

- The graph is connected.
- Each vertex has degree at most $(m-1)n$.
- Each vertex is the intersection of n distinct m-cliques. That is, given any vertex v there exist n disjoint sets of vertices $D_1, D_2, ..., D_n$ so that for each i, $|D_i| = m-1$ and the induced subgraph on $D_i \cup \{v\}$ is an m-clique. Moreover, there are no edges between D_i and D_j for $i \neq j$.
- For $n \geq 4$, no induced subgraph is an n-cycle.

Note that, with the exception of connectedness, these properties are all elementary. It is easy to construct an isomorphism between any two graphs possessing all four of these properties. We call the unique (up to isomorphism) graph with these properties the (m, n) *clique-tree*. A *clique-tree* is simply an (m, n) clique-tree for some m, n. A more concise description of clique-trees was given in [8]: let X, Y

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[1] Since the graph theory involved in this article is combinatorial rather than algebraic, this is our interpretation of the Robinson's message written on p.202 in [10].

be the bipartition of the semi-regular tree $T_{m,n}$ such that every vertex in X has valency m and every vertex in Y has valency n. Then the (m, n) clique-tree is the graph with vertex set Y and two vertices are adjacent if and only if they are of distance 2 in $T_{m,n}$.

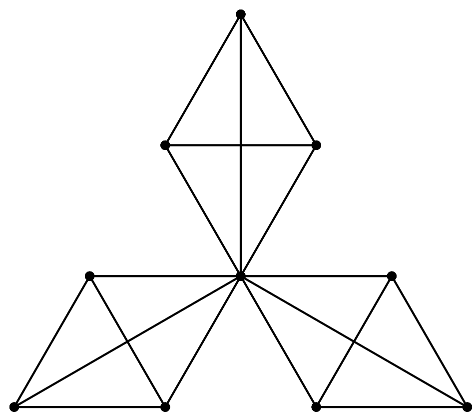

FIGURE 1. A portion of the (4,3) clique-tree

A graph is *ultra-homogeneous* if any isomorphism between finite induced subgraphs extends to an automorphism. Finite ultra-homogeneous graphs were classified by Gardiner in [6, Theorem 12].

Theorem 1.1. *A finite graph is ultra-homogeneous if and only if it is one of the following:*

- *a finite disjoint union of isomorphic finite complete graphs;*
- *a regular complete t-partite graph $K_{t;r}$, $t \geq 2$ and r is the regularity.*
- *a 5-cycle;*
- *the line graph of $K_{3,3}$.*

We would also like to mention that for countable ultra-homogeneous graphs, the isomorphism types were characterized by Lachlan and Woodrow in [11].

An *isometry* is a map that preserves distance. A graph is *2-transitive* if any isometry between pairs of vertices extends to an automorphism. A graph is *finitely transitive* if any isometry between finite tuple of vertices extends to an automorphism. Finitely transitive finite graphs were classified by Cameron in [2].

Theorem 1.2. *A finitely transitive graph must be one of the following:*

- *connected finite ultra-homogeneous graph;*
- *for each n:*
  - *the n-cycle;*
  - *the $(n-1)$-regular bipartite graph with two sets of size n;*
- *the icosahedron;*
- *$J(6, 3, 2)$ (a Johnson graph).*

Remarkably, MacPherson proved that, among connected infinite locally finite graphs, the clique-trees are precisely those graphs that are 2-transitive [13]. Using this result, the following theorem was proved in [9].

Theorem 1.3. *Let G be a connected infinite locally finite graph. Then the following are equivalent:*

(1) G is 2-transitive;
(2) G is finitely transitive;
(3) G is a clique-tree.

2. Quantifier-eliminable Locally Finite Graphs

Let L be a first-order signature. An L-structure is *quantifier-eliminable* if its complete L-theory admits quantifier elimination in L. An L-structure is *ultra-homogeneous* if every isomorphism between finitely-generated substructures extends to an automorphism of M; and we use the term *ultra-homogeneous in L* if we want to emphasize the signature. As was shown in [10, Corollary 7.4.2], these two notions are equivalent on finite structures.

Proposition 2.1. *Let M be a finite structure. Then M is quantifier-eliminable if and only if M is ultra-homogeneous.*

We give two easy applications of Proposition 2.1. Let L_1 be the first-order signature consisting of only one binary relation symbol d_1. We view a graph as an L_1-structure by interpreting d_1 as the edge relation. It is clear that for graphs, ultra-homogeneous means ultra-homogeneous in L_1. It follows that a finite graph is quantifier-eliminable in L_1 if and only if it is isomorphic to a graph listed in Theorem 1.1.

Let L_∞ be the signature of distance predicates, i.e. the signature obtained by adding to L_1, binary relation symbols $d_2, d_3, \ldots$. We view graphs as L_∞-structures by requiring d_k holds for two vertices if and only if they are distance k apart. It is clear that for graphs, finitely transitive means ultra-homogeneous in L_∞. It follows that a finite graph is quantifier-eliminable in L_∞ if and only if it is isomorphic to a graph listed in Theorem 1.2.

The following quantifier-elimination test extends Proposition 2.1. It fits particularly well into our combinatorial setting. We found it as an exercise in [10, Q.6 p.207] but decided to include a proof here since we cannot find a convenient reference.

Proposition 2.2. *Let L be a first-order language and T a complete theory in L. Then T has quantifier elimination if and only if every model of T has an ultra-homogeneous elementary extension.*

Proof. If T has a finite model, then this is a consequence of Proposition 2.1; and so, we assume T has no finite models.

Suppose T does not have quantifier elimination. Then there exist tuples $\bar{a}$ and $\bar{b}$ in a model M of T such that $\bar{a}$ and $\bar{b}$ have the same quantifier-free type but not the same type in M. Because they have the same quantifier-free. type, there is a partial isomorphism sending $\bar{a}$ to $\bar{b}$. Because they do not have the same type, this partial isomorphism does not extend to an automorphism of M nor to an automorphism of any elementary extension of M.

Conversely, suppose that T has quantifier elimination. By Proposition 2.2.7 of [1], M has a strongly ω-homogeneous elementary extension, say N. We argue that N is ultra-homogeneous. Let f be an isomorphism between two finitely generated substructures A and B of N. Suppose $\bar{a}$ generates A and so $\bar{b} := f(\bar{a})$ generates B.

Since f is an isomorphism between substructures, $\bar{a}$ and $\bar{b}$ have the same quantifier-free type in N. Since $N \models T$ and T admits quantifier elimination $\bar{a}$ and $\bar{b}$ have the same type in N. By strong ω-homogeneity, there is an automorphism of N sending $\bar{a}$ to $\bar{b}$ hence extending f. □

We will generalize Proposition 2.1 to locally finite graphs. However, the proofs involved are not harder for a more general class of relational structures, so we will argue in that generality.

3. Ultra-homogeneous relational structures

First, we introduce several notions that we need. Let M be a relational structure, i.e. an L-structure for some first-order signature L which consists of relation symbols only. For any $R \in L$, we say that two distinct elements a, b of M are *R-related* if R^M holds for some tuple of M which contains both a and b. In this case, we also say that a and b are *R-neighbors*. The *Gaifman graph* of M, denoted by $G(M)$, is the graph with vertices elements of M and there is an edge between two vertices if and only if they are R-related for some $R \in L$. For $k \in \mathbb{N}$, the *k-neighborhood* of $a \in M$, denote by $N_k(a)$, is the substructure of M whose underlying set is the set of all vertices with distance at most k from a in $G(M)$. The *component of a* is the substructure $\bigcup_{k \geq 1} N_k(a)$. A *component* of M is the component of its elements. Clearly, M is the disjoint union of its components. The reader can consult [5] for more details about Gaifman graphs.

We say that M is *L-locally finite* or *locally finite in L* if each element of M has finitely many R-neighbors for each $R \in L$. This generalizes the notion of locally finite in the graph-theoretical sense to arbitrary relational structures. For graphs, locally finite, locally finite in L_1, and locally finite in L_∞ are all equivalent. The notion of L-locally finite is not to be confused with the notion of locally finite in the model-theoretic sense. Because finitely generated substructures of a relational structure are necessarily finite, all relational structures are locally finite in the model-theoretic sense. An L-theory is *uniformly L-locally finite* if for each $R \in L$, there exists a natural number $n(R)$ (depends only on R) such that "for each x there exist at most $n(R)$ y's that are R-related to x" is expressed by a sentence in the theory. An L-structure is *uniformly L-locally finite* if its L-theory is. This is not to be confused with the similarly named concept called "uniformly locally finite" (c.f. [10, p.175]).

Lemma 3.1. *If M is ultra-homogeneous then so are its components. Moreover if elements of M are isomorphic as substructures then components of M are isomorphic.*

Proof. Let f be an isomorphism between two finitely generated substructures A and B of a component M_0 of M. By ultra-homogeneity, f extends to an automorphism σ of M. Since σ must send the component of A to the component of B, it restricts to an automorphism of M_0. Moreover if elements of M are isomorphic as substructures, M must be vertex-transitive (i.e. its automorphism group acts transitively on its elements) and hence has isomorphic components. □

Theorem 3.2. *Suppose M is an L-locally finite ultra-homogeneous structure with isomorphic elements. Then any L-structure that is elementarily equivalent to M is a disjoint union of copies of a connected ultra-homogeneous L-structure.*

Proof. By Lemma 3.1, components of M are isomorphic copies of a connected ultra-homogeneous structure, say M_0. Suppose M' is elementarily equivalent to M, it suffices to show that components of M' are isomorphic to M_0.

It follows from L-locally finiteness of M that for every $k \in \mathbb{N}$, $N_k(a)$ is a finite structure for any $a \in M$. Since M is vertex transitive, these k-neighborhoods are all isomorphic a finite L-structure, say N_k. Since N_k is finite, the fact every element has N_k as its k-neighborhood is described in the L-theory of M. This completes the proof since a component of an L-structure is the union over k of these k-neighborhoods. □

For infinite structures, homogeneity is related to strong minimality. Recall that an infinite L-structure M is said to be *strongly minimal* if every definable subset of every structure M' elementarily equivalent to M is either finite or co-finite. By *definable*, we mean definable by an L-formula with parameters from M'. If $D \subset M'$ is defined by an L-formula having parameters in $A \subset M'$, then we say that D is defined *over* A. If $d \in M'$ is an element of a finite set definable over A, then d is said to be *algebraic* over A. Otherwise, d is *non-algebraic* over A. We say that A is *algebraically closed* in M' if every element $d \in M'$ that is algebraic over A is an element of A.

A structure M is *homogeneous* (in the model-theoretic sense) if for each $A \subset M$ with $|A| < |M|$ and each $c \in M$, every elementary map on A can be extended to an elementary map on $A \cup \{c\}$.

Lemma 3.3. *Strongly minimal structures are homogeneous.*

Proof. Let M be strongly minimal and $f\colon A \to M$ be a partial elementary map where $A \subset M$ and $|A| < |M|$. Pick $c \in M \setminus A$. Without loss of generality, we can assume both A and $f(A)$ are algebraically closed in M (see [10, Lemma 9.2.5] for example). Thus the type of c over A and hence its image under f are non-algebraic. Since M is strongly minimal, the latter must be the unique non-algebraic type over $f(A)$ and hence satisfied by any $d \in M \setminus f(A)$. Sending c to d extends f to an elementary map on $A \cup \{c\}$. Thus M is homogeneous. □

If we further assume the structure is quantifier-eliminable then ultra-homogeneity follows.

Lemma 3.4. *Strongly minimal structures that are quantifier-eliminable must be ultra-homogeneous.*

Proof. By Lemma 3.3, the structure is homogeneous. By quantifier elimination any isomorphism between (finitely-generated) substructures is partial elementary and hence is a restriction of an automorphism [14, Proposition 4.2.13]. □

The next two lemmas bring L-locally finiteness into the picture.

Lemma 3.5. *Let M be an infinite quantifier-eliminable L-structure. Suppose that the elements of M are isomorphic as L-structures. Then M is strongly minimal if it is uniformly L-locally finite.*

Proof. Let M' be an elementary extension of M. Let $\phi(x)$ be an L-formula with parameters in M'. By quantifier elimination, we may assume that $\phi(x)$ is a Boolean combination of atomic formulas. To show that $\phi(x)$ defines a finite or cofinite subset M', it suffices to show that each atomic formula $\alpha(x)$ defines a finite or cofinite set. If $\alpha(x)$ has parameters in M', then this follows from the assumption that M is uniformly L-locally finite. Otherwise, if $\alpha(x)$ has no parameters, then it follows from the assumption that elements of M are isomorphic as L-structures that $\alpha(x)$ or its negation defines the empty set. □

We thank Dugald MacPherson for pointing out a mistake in our original version of Lemma 3.5. We left out the assumption that elements of M are isomorphic as L-structures. Without that the statement is simply wrong: for example, suppose L consists of only a unary predicate P and M is an infinite L-structure in which P picks out an infinite co-infinite set. Then clearly M is L-locally finite but not strongly minimal. In addition, we remark that instead of assuming points are isomorphic, Lemma 3.5 holds under a weaker assumption that quantifier free 0-definable sets are either finite or cofinite. However, we state Lemma 3.5 in its current form since we do assume points are isomorphic in the situation where it is applied (Theorem 4.1).

Lemma 3.6. *Suppose M is L-locally finite, quantifier-eliminable and elements of M are isomorphic as L-structures then M is uniformly L-locally finite.*

Proof. By Proposition 2.2, M has an ultra-homogeneous elementary extension M'. Since elements of M are isomorphic as L-structures, the automorphism group of M' act transitively on them. Thus for each $R \in L$, the elements of M must have the same number of R-neighbors. This common value is finite, since M is L-locally finite, and hence must be encoded in the L-theory of M. □

4. Conclusion

Theorem 4.1. *Suppose M is L-locally finite and elements of M are isomorphic as substructures then the following are equivalent.*

(1) M is quantifier-eliminable in L;
(2) M is ultra-homogeneous in L;
(3) There exists a connected ultra-homogeneous L-structure M_0 so that every L-structure elementarily equivalent to M is a disjoint union of copies M_0.
(4) Every L-stricture elementarily equivalent to M is ultra-homogeneous in L.

Proof. By Proposition 2.1, the first two conditions are equivalent for finite structures; so we assume M is infinite. Suppose M is quantifier-eliminable in L, then M is uniformly L-locally finite by Lemma 3.6. It then follows from Lemma 3.5 that M is strongly minimal and hence ultra-homogeneous by Lemma 3.4. This completes the proof that (1) implies (2). (It also shows that (1) implies (4).)

By Theorem 3.2, (2) implies (3).

Now suppose (3) holds and let M' be elementarily equivalent to M. Let f be an isomorphism the substructures of M' generated by two finite tuples $\bar{a}$ and $\bar{b}$, respectively. By reordering the tuples if necessary, we can assume $\bar{a} = \bar{a}_1, \ldots, \bar{a}_k$ and a_i's coming from distinct components. Let A_i $(1 \le i \le n)$ be the component of a_i and B_i $(1 \le i \le n)$ be the component of $\bar{b}_i := f(\bar{a}_i)$. Note that the B_i's are distinct as well. Because A_i and B_i are isomorphic to M_0, there is an isomorphism $g_i \colon A_i \to B_i$ for each $1 \le i \le n$. Let $\bar{c}_i$ be the tuple from A_i such that $g_i(\bar{c}_i) = \bar{b}_i$. By assumption, A_i is ultra-homogeneous so there exists σ_i, an automorphism of A_i, mapping $\bar{a}_i$ to $\bar{c}_i$. Thus $g_i \circ \sigma_i$ is an isomorphism from A_i to B_i that maps $\bar{a}_i$ to

$\bar{b}_i$. Therefore, $h \cup \bigcup_{i=1}^k g_i \circ \sigma_i$ is an automorphism of M' extending f where h is an isomorphism between $M' \setminus \bigcup_{i=1}^k A_i$ and $M' \setminus \bigcup_{i=1}^k B_i$. This shows (3) implies (4).

Lastly, (4) implies (1) by Proposition 2.2. □

Since vertices of a graph are isomorphic as substructures. It follows immediately from Theorem 4.1 and graph theoretic results Theorem 1.2 and 1.3 that

Theorem 4.2. *For a locally finite graph G, the following are equivalent.*

(1) G is quantifier-eliminable in L_∞.
(2) G is the disjoint union of isomorphic copies of
- *one of the finite graphs listed in Theorem 1.2; or*
- *a clique-tree.*

We conclude this article by showing that the complete n-ary tree ($n \geq 1$) is quantifier-eliminable in the signature $L_\infty(r) := L_\infty \cup \{r\}$ where r is a constant symbol. Let T_n be the complete n-ary tree, i.e. the connected acyclic graph with a distinguish vertex called root of valency n and every other vertex has valency $n+1$. We view T_n as an L_∞-structure by interpreting r as the root and d_k ($k \geq 1$) as the distance k relation.

Theorem 4.3. T_n ($n \geq 1$) is quantifier-eliminable in $L_\infty(r)$.

Proof. By Proposition 2.2, it suffices to show that T_n is ultra-homogeneous in $L_\infty(r)$. Let T^0, T^1 be two copies of T_n with roots r_0, r_1, respectively. Let G be the graph obtained by joining T^0 with T^1 at their roots by an edge. Then G is the $(2, n+1)$ clique-tree. Suppose f is an $L_\infty(r)$-isomorphism between two finitely generated $L_\infty(r)$-structure of T^0. Then in particular f is an isomorphism between finitely generated L_∞-substructures of G. Hence f extends to an L_∞-automorphism σ by ultra-homogeneity of G. Since σ fixes r_0, if it maps any $v \in T^0$ to a vertex in T^1 then, by considering distances, σ has to map every vertex on the unique path between v and r_0 into T^1. That means σ has to map a neighbor of r_0 to r_1. Moreover, again by considering distances, if σ maps a vertex of T^0 into T^1, then it has to map the whole subtree below that vertex into T^1. Thus σ at worst swaps a subtree of T^0 with root a neighbor of r_0 with T^1. Fixing the rest while swapping these subtrees again if necessary, we get an L_∞-automorphism of G fixing r_0 that restricts to an automorphism of T^0. This concludes the proof. □

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