ON A RESULT OF ROSENLICHT

WAI YAN PONG

ABSTRACT. We prove a model theoretic result about orthogonality in the theory of differentially closed fields. Using that, we deduce a result of Rosenlicht (Proposition 1).

The following proposition [Ma, Theorem 6.12], usually credited to Rosenlicht, shows that there are no new constants in certain extensions of differential fields.

Proposition 1 (Rosenlicht). *Let $(\mathcal{F}, D)$ be a differential field of characteristic* 0 *such that its field of constants, $\mathcal{C} = \mathcal{C}_{\mathcal{F}}$, is algebraically closed. Let $f(z) \in \mathcal{C}(z)$ and let a be a solution of the differential equation $Dz = f(z)$, where a is transcendental over $\mathcal{F}$. Suppose that $\frac{1}{f(z)}$ is not of the form $c\frac{du}{dz}/u$ or $c\frac{dv}{dz}$ for any u or $v \in \mathcal{C}(z), c \in \mathcal{C}$. Then $\mathcal{C}_{\mathcal{F}(a)} = \mathcal{C}$.*

This result has an interesting application in model theory. It was used (together with other results) in showing that the theory of differentially closed fields has the dimension order property [Ma, Theorem 7.4]. It then follows that for any uncountable cardinal κ there are 2^{κ} non-isomorphic differential fields of size κ.

The main result of this article, Theorem 2.1, is stated in the paper [H-I] by Hrushovski and Itai. They also indicated (see Remark 1.16 [H-I]) the possibilities of proving Theorem 2.1 in a similar way as Lemma 1.15 in [H-I]. Here we simply carry out what they suggested and observe that Proposition 1 in turn can be deduced from Theorem 2.1. We hope the proofs given here will provide an interesting point of view from Model Theory.

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1. STRONGLY MINIMAL SETS ON CURVES

All the materials in this section, except the examples, are in Section 1 of [H-I]. Here we simply give a brief account of the ideas and results that we need subsequently. So we will assume some familiarity of differential algebra and refer our readers to [Ko] for the concepts and notations that we use without giving the definitions.

Let (K, D) be a differentially closed field of characteristic 0. Let X be a irreducible Kolchin closed set of finite dimension. i.e. The field of differential

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rational functions on X, $K\langle X\rangle$, has finite transcendence degree over K. Thus if a is the generic point of X over K, then $K\langle X\rangle = K(a, Da, \ldots D^n a)$ for some $n \geq 0$. If V is an algebraic variety over K with function field $K\langle X\rangle$, then we say that X *lives birationally (or simply lives) on* V. For instance, one such V will be the Zariski closure over K of the point $(a, Da, \ldots D^n a)$. So after a birational change, X can be identified with the subset of V given by

$$\{v \in V : Dv = t(v)\}$$

where t is a rational map on V. By going to smaller open sets, we can further assume V is nonsingular and t is a regular map on V. Let us also point out that if X is a Kolchin closed set living on V then any Kolchin closed $X' \subset X$ is of the form

$$X' = \{v \in V' : Dv = t(v)\}$$

with V' a Zariski closed subset of V (see [H-I, Remark 1.1]). So in the case where V is a curve then any Kolchin closed set living on V is strongly minimal. Let us give two examples to illustrate these ideas.

Example 1.1. Consider the differential polynomial $A(y) = (Dy)^2 - y^3$. Since A is irreducible as a polynomial, we have $P = \{A\} : S_A = \{A\} : Dy$ is a prime differential ideal. Let X be the irreducible Kolchin closed set corresponding to P. Then $2D^2y = 3y^2$ in $K\{X\}$. Hence X lives on the variety V defined by $x_1^3 - x_2^2$ and X is defined by $D(x_1, x_2) = (x_2, \frac{3}{2}x_1^2)$ on V.

Example 1.2. Consider the differential polynomial $A(y) = yD^2y - Dy$. Similarly let X be the irreducible Kolchin closed set correspond to $\{A\} : S_A$. Then X lives on $\mathbb{A}^2$ and defined by $D(x_1, x_2) = (x_2, \frac{x_2}{x_1})$.

From now on, we will assume K is sufficient saturated, with k the field of constants. Let k_0 be a small algebraically closed subfield of k. Let V be a variety defined over k_0. In this case, the first prolongation of V is the tangent bundle $\pi : TV \to V$. And we have a natural differential section, ∇, of π given generically by $a \mapsto (a, Da)$. So a Kolchin closed set living on V has the form

$$\{v \in V : \nabla(v) = \sigma(v)\}$$

where σ is some rational section of π. In the case where V is a curve, if σ is the zero section then X is simply $V(k)$. Otherwise the equation $\omega(v)(\sigma(v)) = 1$ defines a rational 1-form ω on V. We set

$$X(V, \omega) = \{v \in V : \omega(v)Dv = 1\}.$$

We declare that a pole c of ω is in $X(V, \omega)$ if and only if $Dc = 0$. Writing ω in terms of local coordinates, one sees immediately that this condition is forced if we want $X(V, \omega)$ to be Kolchin closed. We summarize the above discussion in the following lemma:

Lemma 1.3. *Let Y be any strongly minimal set defined over the constants living on a curve C. Then, up to a finite set, either $Y = C(k)$ (so Y is defined by $Dx = 0$) or $Y = X(C, s)$ for some rational 1-form s.*

From now on, unless otherwise indicated, all the objects in our discussion are defined over k_0. Also it is understood that a 1-form is a rational 1-form.

Lemma 1.4. *Let C be a curve and $g \in k(C) \setminus k$. Let s be a 1-form on C. Then $gX(C, s)$ is not almost contained in the constants.*

Proof. Since $\dim \Omega(C)$ over $k(C)$ is 1. We can write $dg = hs$ for some $h \in k(C)$. If $h = 0$ then g is a constant function contradicting our assumption. Otherwise, for $c \in X(C, s)$

$$Dg(c) = dg(c)Dc = h(c)s(c)Dc = h(c).$$

Thus if $gX(C, s)$ is almost contained in the constants (i.e. after taking out finitely many points, the points in $gX(C, s)$ are all constant), then h vanishes on $X(C, s)$ minus finitely many points which is still infinite. Hence $h = 0$, so again is a contradiction. □

Lemma 1.5. *Let C be a curve and s be a nonzero 1-form on C such that $X(C, s)$ lives on C. Then $X(C, s)$ is almost orthogonal to the constants.*

Proof. Suppose not then there exist $a \in X(C, s)$ and a constant c such that $c \in \mathrm{acl}(k_0(a)) \setminus k_0$. By elimination of imaginaries (in DCF_0), S, the set of conjugates of c over $\mathrm{acl}(k_0(a))$ is coded by a tuple $(t_1, \ldots, t_m)$ of elements of k. Fixing $k_0(a)$ pointwise will fix S set-wise and hence its code. So each t_i is definable over $k_0(a)$. But at least one of them, call it t, is transcendental over k_0, otherwise c is algebraic over k_0, contradiction. Since $X(C, s)$ is Zariski dense in C, t defines a rational function on C (still call it t) with $tX(C, s) \subset k$. But this contradicts Lemma 1.4. □

We will also need the following results to prove Theorem 2.1. We refer our readers to [H-I] for their proofs.

Lemma 1.6. *Let X be a Kolchin closed strongly minimal set living on a variety V. Let E be a definable equivalence relation on X with finite classes. Then there exists a variety U, rational map $f : V \to U$ and a Kolchin-closed strongly minimial $Y \subset U$, such that Y lives on U, and for almost all $a \in X, f(a) \in Y$, and $f^{-1}(f(a))$ is the E-class of a.*

Lemma 1.7. *Let $g : C_1 \to C_2$ be a dominant rational map between nonsingular curves defined over k. Let s_2 be a 1-form on C_2, defines over k_0, and let $s_1 = g^* s_2$ be the pullback of s_2 by g. Then $g^{-1}X(C_2, s_2) = X(C_1, s_1)$.*

Lemma 1.8. *Let $g : C_1 \to C_2$ be a dominant rational map between nonsingular curves. Let s_i be a 1-form on C_i. Assume $gX(C_1, s_1) \subset X(C_2, s_2)$ (up to a finite set.) Then $s_1 = g^* s_2$.*

2. Orthogonality to the constants

A minor modification of the proof of Lemma 1.15 in [H-I] gives:

Theorem 2.1. *Let C be a curve and s be a rational 1-form on C. Then*

$$X(C,s) \not\perp k$$

if and only if there is a dominant rational map

$$g: C \to A = \mathbb{G}_a, \mathbb{G}_m \text{ or an elliptic curve}$$

such that $s = g^*\omega$ for some global invariant 1-form ω on A.

Proof. For one direction, suppose $g: C \to A$ and ω are given as in the statement of the theorem. By Lemma 1.7, $gX(C,s) = X(A,\omega)$. For any $a \in A(k)$, denote by m_a the left multiplication by a. Since ω is an invariant form, we have $m_a X(A,\omega) = X(A, m_a^*\omega) = X(A,\omega)$. Together with the facts that $X(A,\omega)$ is Kolchin closed and strongly minimal, $X(A,\omega)$ must be a coset of $A(k)$ and hence $X(C,s)$ is non-orthogonal to the constants.

For the other direction, suppose $X(C,s) \not\perp k$ but by Lemma 1.5, it is almost orthogonal to k. So by general results on almost orthogonality [Hr], we have

1. (1) An equivalent relation E on $X(C,s)$ with finite classes.
2. (2) A definable transitive action of a connected definable group G on $X(C,s)/E$.

So by Lemma 1.6, there is a smooth projective curve A, a dominant rational map $g: C \to A$ and a Kolchin closed strongly minimal set Y that lives on A such that for almost all $a \in X$, we have $g(a) \in Y$ and $g^{-1}(g(a)) = E_a$. By Lemma 1.3 and Lemma 1.4, Y is of the form $X(A,\omega)$ for some rational 1-form ω on A. By Lemma 1.8, after throwing away a finite subset of $X(A,\omega)$, we can assume $gX(C,s) = X(A,\omega)$ and $s = g^*\omega$. By (2) the action of each $\alpha \in G$ is definable hence coincides with a rational function f_α on $X(A,\omega)$. Since $X(A,\omega)$ is Zariski dense in A, f_α extends to a rational function (still call it f_α) from A to itself. By considering $\alpha^{-1} \in G$, we see that f_α must be an isomorphism of A. So we have $X(A,\omega) = f_\alpha^{-1}X(A,\omega) = X(A, f_\alpha^*\omega)$ and therefore $\omega = f_\alpha^*\omega$. Let F be the set of zeros and poles of ω. It is clear that each f_α fixes F setwise. Note also that since G acts transitively on an infinite set $X(A,w)$, the collection $\{f_\alpha\}_{\alpha \in G}$ must be infinite. However in characteristic 0, if the genus of $A > 1$, then $\mathrm{Isom}(A)$, the group of isomorphisms from A to itself is finite (see [Ha, p.305, 2.5]). Thus we are left with the genus 1 and genus 0 case.

Genus of $A = 1$. In this case, A is an elliptic curve. We identify $A \times \mathrm{Aut}(A)$ with $\mathrm{Isom}(A)$ by $(p,\varphi) \mapsto \tau_p \circ \varphi$ where τ_p is translation by p map. Since $\mathrm{Aut}(A)$ (the subgroup of $\mathrm{Isom}(A)$ consists of group automorphisms of A) is finite [Si, Theorem 10.1], any subset of $\mathrm{Isom}(A)$ fixing a finite subset of A has to be finite. Therefore ω must have no poles or zeroes. Hence ω is a global invariant 1-form on the A.

Genus of $A = 0$. In this case, $A = \mathbb{P}^1$. Since any automorphism of $\mathbb{P}^1$ fixing three points is the identity, there are only finitely many automorphisms of $\mathbb{P}^1$ fixing a set with at least three points. So F has at most two points. However any rational 1-form on $\mathbb{P}^1$ will have two more poles than zeros. So ω has exactly two poles and no zeros. One case is that ω has a double pole. Up to an isomorphism of $\mathbb{P}^1$, the pole is at ∞. And since a 1-form on $\mathbb{P}^1$ is completely determined by its poles and zeros. Therefore ω must be of the form cdz ($c \in k_0$). Taking away ∞, ω is a global invariant form on $\mathbb{G}_a$. The other case is that ω has two distinct poles. Again we can assume they are 0 and ∞. So $\omega = c\frac{dz}{z}$ for some c. After taking out these points. ω is a global invariant form on $\mathbb{G}_m$.

This concludes the proof. □

We now deduce the Proposition from Theorem 2.1.

Proof of the Proposition 1. Let b realize a nonforking extension of $\text{tp}(a/\mathcal{F})$ to $\tilde{\mathcal{F}}$, then b is a solution to the original equation and b is transcendental over $\tilde{\mathcal{F}}$. If we can show that $\tilde{\mathcal{F}}(b)$ has the same field of constants as $\mathcal{F}$ then the same is true for $\mathcal{F}(b)$. But the map sending a to b extends to an isomorphism between $\mathcal{F}(a)$ and $\mathcal{F}(b)$ over $\mathcal{F}$, hence $\mathcal{C}_{\mathcal{F}(a)} = \mathcal{C}_{\mathcal{F}}$ as well. So we assume a is transcendental over $\tilde{\mathcal{F}}$.

Let X be the subset in $\mathbb{A}^1$ defined by $Dz = f(z)$. In our notation $X = X(\mathbb{P}^1, \frac{1}{f(z)}dz)$. Since $\mathcal{C}$ is algebraically closed, $\mathcal{F}(a)$ has no new constants is equivalent to X is orthogonal to the constants.

By Hurwitz formula, there are no dominant rational maps from $\mathbb{P}^1$ to an elliptic curve. Apply Theorem 2.1 to the case $k_0 = \mathcal{C}$ and we conclude that $X \not\perp k$ if and only if s is the pullback of either:

(1) cdx under $v : \mathbb{P}^1 \to \mathbb{G}_a$, or
(2) $c\frac{dx}{x}$ under $u : \mathbb{P}^1 \to \mathbb{G}_m$

In the first case, we have $\frac{1}{f(z)}dz = s(z) = v^*(cdx) = dcv(z) = cv'dz$ and hence $\frac{1}{f(z)} = cv'(z)$. In the second case, a similar computation shows that $\frac{1}{f(z)} = c\frac{u'(z)}{u(z)}$. This completes our proof. □

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University of Illinois at Urbana-Champaign, 273 Altgeld Hall, 1409 West Green Street, Urbana, IL 61801.