Some applications of ordinal dimensions to the theory of differentially closed fields

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Abstract

Using the Lascar inequalities, we show that any finite rank δ-closed subset of a quasiprojective variety is definably isomorphic to an affine δ-closed set. Moreover, we show that if X is a finite rank subset of the projective space $\mathbb{P}^n$ and $\mathbf{a}$ is a generic point of $\mathbb{P}^n$, then the projection from $\mathbf{a}$ is injective on X. Finally we prove that if $\mathrm{RM} = \mathrm{RC}$ in DCF_0, then $\mathrm{RM} = \mathrm{RU}$.

We consider the theory of differentially closed fields of characteristic 0 in, $\mathfrak{L}_\delta = \{0, 1, +, \cdot, \delta\}$, the language of rings together with an extra unary function symbol δ for the derivation. Let DCF_0 denote this complete theory. Since we consider only a single derivation δ, we will call a differential field a δ-field. Similar abbreviations apply to other differential objects. *Throughout this paper, $\mathcal{U}$ is a sufficiently saturated differentially closed field of characteristic 0. All the fields that appear are δ-subfields of $\mathcal{U}$. From now on until the end of section one, $\mathcal{K}$ is an ω-saturated differentially closed field. All the affine and projective spaces considered are over $\mathcal{K}$.*

Let us summarize some of the definitions and facts that we are going to use in this paper. In DCF_0, there is an algebraic characterization of forking. Let $\mathcal{G}, \mathcal{H} \supseteq \mathcal{F}$ be δ-fields. We say that $\mathcal{G}$ **does not fork from $\mathcal{H}$ over $\mathcal{F}$**, write as $\mathcal{G} \perp_{\mathcal{F}} \mathcal{H}$, if $\mathcal{G}$ and $\mathcal{H}$ are algebraically disjoint (as fields) over $\mathcal{F}$. Let A, B and C be subsets of $\mathcal{U}$. Later on when we write $A \perp_C B$, we mean $\mathbb{Q}\langle AC\rangle \perp_{\mathbb{Q}\langle C\rangle} \mathbb{Q}\langle BC\rangle$. If C is the empty set, we will simply write $A \perp B$.

With the notion of forking, we can define the **U-rank**, RU, inductively as follows:

Definition Let $\bar{a}$ be a tuple from $\mathcal{U}$ and $B \subset \mathcal{U}$.

- $\mathrm{RU}(\bar{a}/B) \geq 0$.
- If α is a limit ordinal, then $\mathrm{RU}(\bar{a}/B) \geq \alpha$ if $\mathrm{RU}(\bar{a}/B) \geq \beta$, for all $\beta < \alpha$.
- $\mathrm{RU}(\bar{a}/B) \geq \alpha + 1$ if there exists $C \supset B$ such that $\mathrm{RU}(\bar{a}/C) \geq \alpha$ and $\bar{a} \not\downarrow_B C$.

We say that $\mathrm{RU}(\bar{a}/B)$ **is equal to** α if $\mathrm{RU}(\bar{a}/B) \geq \alpha$ but $\mathrm{RU}(\bar{a}/B) \not\geq \alpha+1$.

Let $p \in S_n(A)$ be an n-type over A. The **U-rank of** p, $\mathrm{RU}(p)$, is defined to be $\mathrm{RU}(\bar{a}/A)$ where $\bar{a}$ is any realization of p in $\mathcal{U}^n$. Let B be an A-definable set; **we define** $\mathrm{RU}(B/A)$ **to be** $\sup\{\mathrm{RU}(b/A) : b \in B\}$. In DCF_0 the supremum can be attained; i.e. there exists $b \in B$ such that $\mathrm{RU}(b/A) = \mathrm{RU}(B/A)$. We refer the readers to [12] for a proof. Also note that this definition is independent of A. Suppose B is defined over some other A'. Let $C = A \cup A'$, it is clear that $\mathrm{RU}(B/C) \leq \mathrm{RU}(B/A)$. But since every type over A has a nonforking extension to C, thus $\mathrm{RU}(B/C) = \mathrm{RU}(B/A)$. Replacing A by A', we see that $\mathrm{RU}(B/C) = \mathrm{RU}(B/A')$ as well.

The basic geometric objects that we are interested in are the δ-closed subsets of the affine and the projective spaces. A subset X of $\mathbb{A}^n$ is δ-closed if there exist $f_1, \ldots, f_m \in \mathcal{K}\{t_1, \ldots, t_n\}$ such that

$$X = \{\bar{x} \in \mathbb{A}^n : f_1(\bar{x}) = \cdots = f_m(\bar{x}) = 0\}.$$

A subset X of $\mathbb{P}^n$ is δ-closed if the intersection of X and any affine Zariski open subset of $\mathbb{P}^n$ is δ-closed.

Since δ-closed sets are in particular definable sets, therefore U-rank gives a natural notion of dimension on these sets. It enjoys the following nice properties:

- $\mathrm{RU}(a/B) = 0$ if and only iff a is algebraic over $\mathbb{Q}\langle B\rangle$
- $\mathrm{RU}(a/B) = \omega$ if and only if a is δ-transcendental over $\mathbb{Q}\langle B\rangle$
- RU is invariant under definable bijection.

Moreover, RU satisfies the **Lascar inequalities** which are very useful in computing U-rank.

$$\mathrm{RU}(\bar{a}/A, \bar{b}) + \mathrm{RU}(\bar{b}/A) \leq \mathrm{RU}(\bar{a}, \bar{b}/A) \leq \mathrm{RU}(\bar{a}/A, \bar{b}) \oplus \mathrm{RU}(\bar{b}/A)$$

where $+$ is the usual ordinal sum and $\oplus$ is the Cantor's symmetric sum of ordinals. For a proof, see [7] or [13].

As an example, let us calculate $\mathrm{RU}(\mathbb{P}^n)$ and $\mathrm{RU}(\mathbb{A}^n)$. As a set, $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ where $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n$ for $i = 0, \ldots, n$. It is not hard to see that $\mathrm{RU}(\mathbb{P}^n) = \mathrm{RU}(\mathbb{A}^n)$. Since $\mathcal{K}$ is ω-saturated, there is an $a \in \mathbb{A}^1$ which is δ-transcendental over $\mathbb{Q}$. We have $\mathrm{RU}(a) = \mathrm{RU}(\mathbb{A}^1) = \omega$. By induction hypothesis, there exists $(a_1, \ldots, a_{n-1}) \in \mathbb{A}^{n-1}$ such that $\mathrm{RU}(\mathbb{A}^{n-1}) = \mathrm{RU}(a_1, \ldots, a_{n-1}) = \omega(n-1)$. Again by saturation, there is an $a_n \in \mathbb{A}^1$ such that a_n is δ-transcendental over $\mathbb{Q}\langle a_1, \ldots, a_{n-1}\rangle$. Since $\omega + \omega(n-1) = \omega n = \omega \oplus \omega(n-1)$, both ends of the Lascar inequalities are equal. Therefore $\mathrm{RU}(\mathbb{A}^n) = \omega n$.

Finally we list the facts about DCF_0 that we will use. They can be found in [9] and [11].

1. DCF_0 has quantifier elimination in $\mathfrak{L}_\delta$.

2. DCF_0 has elimination of imaginaries in $\mathfrak{L}_\delta$.

3. DCF_0 is ω-stable.

4. Let $X \subset \mathcal{U}^n$ be a set definable in the pure language of rings (with parameters). Let $\mathrm{RU}^-(X)$ (respectively $\mathrm{RM}^-(X)$) be the U-rank (respectively the Morley rank) of X calculated as a definable subset of $\mathcal{U}^n$ when we regard $\mathcal{U}$ simply as an algebraically closed field (forgetting the differential field structure on $\mathcal{U}$). Then we have $\mathrm{RU}(X) = \omega\,\mathrm{RU}^-(X) = \omega\,\mathrm{RM}^-(X) = \mathrm{RM}(X)$. Moreover, in the case when X is an algebraic variety, we have $\mathrm{RU}^-(X) = \mathrm{RM}^-(X) = \dim(X)$.

Other basic model-theoretic notions and the definitions of the ranks used here are found in [8]. For the model theory of differentially closed fields, readers can consult [9].

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1 Finite rank subsets in affine and projective spaces

We identify hypersurfaces of degree d in $\mathbb{P}^n$ with points in $\mathbb{P}^N$, $(N = \binom{n+d}{d} - 1)$, by:

$$\mathbf{a} = [\cdots, a_I, \cdots] \in \mathbb{P}^N \longleftrightarrow H_{\mathbf{a}} = \left\{ \mathbf{x} \in \mathbb{P}^n : \sum\nolimits_{|I|=d} a_I x^I = 0 \right\}.$$

First we have the following observation.

Proposition 1.1. *Let X be a definable subset in $\mathbb{P}^n$ of finite U-rank, then any generic hypersurface in $\mathbb{P}^n$ does not meet X.*

Proof. Let $H = \sum a_I x^I$ be a generic hypersurface of degree d in $\mathbb{P}^n$, i.e. $\mathbf{a} = [\cdots, a_I, \cdots]$ is a generic point of $\mathbb{P}^N$. Suppose $\mathbf{x} \in X \cap H$, then using $\mathbf{x}$ we can specify $\mathbf{a}$ in a proper subvariety of $\mathbb{P}^N$ namely the hyperplane defined by $[\cdots, x^I, \cdots]$. Therefore, $\mathrm{RU}(\mathbf{a}/\mathbf{x}) \leq \omega(N-1)$; by the Lascar inequalities we have

$$\mathrm{RU}(\mathbf{a}) \leq \mathrm{RU}(\mathbf{a}, \mathbf{x}) \leq \mathrm{RU}(\mathbf{a}/\mathbf{x}) \oplus \mathrm{RU}(\mathbf{x}) \leq \omega(N-1) \oplus \mathrm{RU}(X) < \omega N.$$

However, this contradicts the fact that $\mathbf{a}$ is a generic point of $\mathbb{P}^N$. Therefore, we conclude that $X \cap H = \emptyset$. □

Before proving the next result, let us consider the following example.

Example 1.2. Let $\mathcal{C}$ be the field of constants in $\mathcal{K}$. Consider the following δ-closed subset of the quasiaffine variety $\mathbb{A}^2 \setminus \{\bar{0}\}$

$$\mathcal{C}^2 \setminus \{\bar{0}\} = \left\{ (x, y) \in \mathbb{A}^2 \setminus \{\bar{0}\} : \delta x = \delta y = 0 \right\}.$$

Note that $\mathcal{C}^2 \setminus \{\bar{0}\}$ is isomorphic to the δ-closed subset of $\mathbb{A}^3$ given by

$$\delta t_1 = \delta t_2 = 0, \quad (t_1 - a t_2) t_3 - 1 = 0$$

where $a \in \mathcal{K} \setminus \mathcal{C}$. However, $\mathbb{A}^2 \setminus \{\bar{0}\}$ is not isomorphic to an affine variety. Let us briefly indicate why. Let $U = \mathbb{A}^2 \setminus \{\bar{0}\}$. The zero set of every non-constant polynomial in $\mathcal{K}[x, y]$ is of codimension 1 in $\mathbb{A}^2$, so it cannot be just the origin. From this we see that $\mathcal{O}(U)$, the ring of regular functions on U is $\mathcal{K}[x, y]$. There is a functorial correspondence between morphisms of affine varieties and homomorphisms of rings of regular functions. So if U is affine, then the identity homomorphism of $\mathcal{K}[x,y]$ will induce an isomorphism between U and $\mathbb{A}^2$ but it is easy to check that the identity map actually induces the inclusion from U to $\mathbb{A}^2$. This shows that U is not isomorphic to any affine variety.

Example 1.2 is actually a special case of:

Theorem 1.3. *Let X be a δ-closed subset of a quasiaffine variety $V \subseteq \mathbb{A}^n$ with $\mathrm{RU}(X) < \omega$. Then X is definably isomorphic to a δ-closed set in $\mathbb{A}^{n+1}$.*

Proof. Let $V = F \cap G$ where F is a Zariski closed set and G is a Zariski open set of $\mathbb{A}^n$. Suppose G is given by $\bigvee_{j=1}^{q} g_j(\bar{t}) \neq 0$, where $g_j \in \mathcal{K}[t_1, \ldots, t_n]$, $j = 1, \ldots, q$. As a δ-closed subset of V, let X be defined by

$$\bigwedge_{i=1}^{p} f_i(\bar{t}) = 0 \wedge \bigvee_{j=1}^{q} g_j(\bar{t}) \neq 0$$

where $f_i \in \mathcal{K}\{t_1, \ldots, t_n\}$, $i = 1, \ldots, p$. Let $G_{\bar{z}}(\bar{t}) = \sum_{j=1}^{q} z_j g_j(\bar{t})$ and

$$S = \{(\bar{c}, \bar{x}) \in \mathbb{A}^q \times X : G_{\bar{c}}(\bar{x}) = 0\}.$$

For any $(\bar{c}, \bar{x}) \in S$, since the $g_j(\bar{x})$'s are not all zero, $\bar{c}$ satisfies a nontrivial linear polynomial with coefficients in the δ-field generated by $\bar{x}$ and, B, the canonical base of X. So we have $\mathrm{RU}(\bar{c}/\bar{x}, B) \leq \omega(q-1)$. Thus by the Lascar inequalities, we have

$$\begin{aligned} \mathrm{RU}(\bar{c}/B) &\leq \mathrm{RU}(\bar{c}, \bar{x}/B) \\ &\leq \mathrm{RU}(\bar{c}/\bar{x}, B) \oplus \mathrm{RU}(\bar{x}/B) \\ &\leq \omega(q-1) \oplus \mathrm{RU}(X) < \omega q. \end{aligned}$$

Since $\mathcal{K}$ is ω-saturated, we can choose $\bar{a} \in \mathbb{A}^q$ such that $\mathrm{RU}(\bar{a}/B) = \omega q$. For such an $\bar{a}$, we will have $(\bar{a}, \bar{x}) \notin S$ for all $\bar{x} \in X$, hence

$$X \subseteq \{\bar{x} \in \mathbb{A}^n : G_{\bar{a}}(\bar{x}) \neq 0\} \subseteq \left\{\bar{x} \in \mathbb{A}^n : \bigvee_{j=1}^{q} g_j(\bar{x}) \neq 0\right\}.$$

Thus X is also given by

$$\bigwedge_{i=1}^{p} f_i(\bar{t}) = 0 \wedge G_{\bar{a}}(\bar{t}) \neq 0.$$

Let $\hat{X}$ be the affine δ-closed set in $\mathbb{A}^{n+1}$ defined by the f_i's and $G_{\bar{a}}(\bar{t})t_{n+1} - 1$. It is easy to see that $\hat{X}$ and X are definably isomorphic via the projection $(a_1, \ldots, a_{n+1}) \to (a_1, \ldots, a_n)$. $\square$

The following is an immediate consequence of Proposition 1.1 and Theorem 1.3.

Corollary 1.4. *Let $V \subseteq \mathbb{P}^n$ be a quasiprojective variety. Let X be a δ-closed subset in V of finite U-rank, then X is definably isomorphic to a δ-closed set in $\mathbb{A}^{n+1}$.*

Proof. By Proposition 1.1, $X \subseteq V \setminus H_{\mathbf{a}}$ where $\mathbf{a}$ is a generic point of $\mathbb{P}^n$. Since $V \setminus H_{\mathbf{a}}$ is quasiaffine, the assertion follows from Theorem 1.3 $\square$

By a result of Pillay ([11], Lemma 1.1) every finite rank δ-group can be embedded into an algebraic group. Since algebraic groups are quasiprojective, Corollary 1.4 yields as a special case the result of Hrushovski and Sokolović's ([3], Lemma 3.3) that any finite rank δ-group is differentially affine.

The results we have obtained so far are based on a simple idea—**sets of finite U-rank are "small" so anything related to them cannot possibly be generic**. Along the same line we prove:

Proposition 1.5. *Let X be a finite rank subset of $\mathbb{P}^n$, with $n \geq 2$. Let $\mathbf{a} \in \mathbb{P}^n \setminus X$ with $\mathrm{RU}(\mathbf{a}) \geq \omega 2$. Then $\pi_{\mathbf{a}}|_X$ is injective where $\pi_{\mathbf{a}}$ is the projection from $\mathbb{P}^n \setminus \{\mathbf{a}\}$ to $\mathbb{P}^{n-1}$ with center at $\mathbf{a}$.*

Proof. Suppose not, then there exist $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{a}$ is on the line determined by $\mathbf{x}$ and $\mathbf{y}$ which is isomorphic to $\mathbb{P}^1$. So we have

$$\mathrm{RU}(\mathbf{a}/\mathbf{x}, \mathbf{y}) \leq \omega.$$

Therefore

$$\begin{aligned} \mathrm{RU}(\mathbf{a}) &\leq \mathrm{RU}(\mathbf{a}/\mathbf{x}, \mathbf{y}) \oplus \mathrm{RU}(\mathbf{x}, \mathbf{y}) \\ &\leq \omega \oplus 2\,\mathrm{RU}(X) \\ &< \omega 2. \end{aligned}$$

This is a contradiction. $\square$

In fact, if we consider the general k-plane in $\mathbb{P}^n$ as the generic point of the Grassmannian, then we get the following result (see Proposition 3.1.5 in [12]).

Proposition 1.6. *Let X be a definable subset of $\mathbb{P}^n$ of finite U-rank, ($n \geq 2$). Let L be a generic k-plane in $\mathbb{P}^n$ and $\pi_L : \mathbb{P}^n \setminus L \to \mathbb{P}^{n-k-1}$ be a projection from L. Then the restriction of π_L to X is injective.*

Using Proposition 1.5, we can show that:

Theorem 1.7. *Any finite U-rank subset of $\mathbb{P}^n$ is definably isomorphic to a definable subset of $\mathbb{A}^1$.*

Proof. By Proposition 1.5, we see that $\pi_{\mathbf{a}}|_X$ has a definable inverse given by $\mathbf{y} \mapsto \overline{\mathbf{a}y} \cap X$. Hence $\pi_{\mathbf{a}} X$ is definably isomorphic to X. So by successively applying the (1.5), we see that every finite rank set in $\mathbb{P}^n$ is definably isomorphic to a subset of $\mathbb{P}^1$. But clearly the image of X is still of finite U-rank, so it is actually sitting inside a copy of $\mathbb{A}^1$. □

So as far as those properties invariant under definable bijection are concerned, one can reduce the study of finite rank sets to the study of finite rank subsets of the affine line.

2 Is Morley rank equal Lascar rank in DCF?

It is well-known that in the theory of algebraically closed field, Morley rank (RM) is equal to U-rank. In DCF_0, as in any ω-stable theory, we know that $\mathrm{RU} \leq \mathrm{RC} \leq \mathrm{RM}$, where RC is the continuous rank (see [8][Chapter 4]). The rank (in any of the above senses) of an element a over a set B is always less than or equal to ω. It is equal to ω if and only if a is δ-transcendental over $\mathbb{Q}\langle B\rangle$. In this chapter, we are going to study the relations between these ranks in DCF_0. We show that if the Morley rank of a type p is a limit ordinal, then $\mathrm{RU}(p) = \mathrm{RM}(p)$. From this we deduce that in DCF_0, $\mathrm{RM} = \mathrm{RC}$ implies $\mathrm{RM} = \mathrm{RU}$. Since our proof, Hrushovski and Scanlon have shown that RM and RU differ in DCF_0 (see Appendix A). Therefore we also have $\mathrm{RM} \neq \mathrm{RC}$ in DCF_0.

2.1 The Kolchin Polynomials

We start by reviewing some notions of differential algebra. Let $\mathcal{R}$ be a δ-ring. Denote by $\mathrm{Spec}_\delta \mathcal{R}$ the set of all prime δ-ideals in $\mathcal{R}$. Together with the induced Zariski topology, $\mathrm{Spec}_\delta \mathcal{R}$ can be regarded as a subspace of $\mathrm{Spec}\mathcal{R}$. Moreover if $\mathrm{Spec}_\delta \mathcal{R}$ is Noetherian, then every prime δ-ideal of $\mathcal{R}$ has a natural ordinal dimension.

Definition 2.1. Let $\mathcal{P} \in \mathrm{Spec}_\delta \mathcal{R}$. The **$\delta$-dimension** of $\mathcal{P}$ is defined inductively as:

- $\dim_\delta \mathcal{P} = 0$ if $\mathcal{P}$ is a maximal element in $\mathrm{Spec}_\delta \mathcal{R}$;
- $\dim_\delta \mathcal{P} = \sup \{\dim_\delta \mathcal{Q} + 1 : \mathcal{Q} \in \mathrm{Spec}_\delta \mathcal{R}, \mathcal{Q} \supsetneq \mathcal{P}\}$.

Furthermore, if $\mathcal{R}$ is also an integral domain, then we define the **δ-dimension of $\mathcal{R}$,** $\dim_\delta \mathcal{R}$ to be the δ-dimension of the zero ideal of $\mathcal{R}$.

Let $\mathcal{K}$ be a differentially closed field. To each type $p \in S_n(\mathcal{K})$, we associate a prime δ-ideal

$$\mathcal{I}_p = \{f(\bar{t}) \in \mathcal{K}\{t_1, \ldots, t_n\} : \text{“}f(\bar{t}) = 0\text{”} \in p\}.$$

This is a one to one correspondence between $S_n(\mathcal{K})$ and $\mathrm{Spec}_\delta \mathcal{K}\{t_1, \ldots, t_n\}$. If $\bar{a}$ is a realization of p, then $\mathcal{I}_p = \mathrm{Ker}\varphi_{\bar{a}}$ where $\varphi_{\bar{a}}$ is the δ-$\mathcal{K}$-algebra homomorphism from $\mathcal{K}\{t_1, \ldots, t_n\}$ to $\mathcal{U}$ determined by sending t_i to a_i, for $i = 1, \ldots, n$.

Morley rank and δ-dimension are related. Our next result shows that Morley rank is bounded above by δ-dimension.

Lemma 2.2. *Let $\mathcal{L}$ be a δ-field. Let $\mathcal{K}$ be an ω-saturated differentially closed extension of $\mathcal{L}$. Let $p \in S_n(\mathcal{L})$ and $\tilde{p} \in S_n(\mathcal{K})$ be a nonforking extension of p on $\mathcal{K}$. Then* $\mathrm{RM}(p) \leq \dim_\delta I_{\tilde{p}}$.

Proof. Since $\tilde{p}$ is a nonforking extension of p, $\mathrm{RM}(p) = \mathrm{RM}(\tilde{p})$. It suffices to show that **for any** $q \in S_n(\mathcal{K})$**, if** $\mathrm{RM}(q) \geq \alpha$ **then** $\dim_\delta \mathcal{I}_q \geq \alpha$. Since δ-dimension of a prime δ-ideal is always nonnegative, the implication clearly holds for the case $\alpha = 0$. When α is a limit ordinal, the implication follows easily from the induction hypothesis. So suppose $\mathrm{RM}(q) \geq \alpha + 1$. Let $f_1, \ldots, f_k$ be a basis of $\mathcal{I}_q$ then the formula $\bigwedge_{i=1}^k f_i(\bar{t}) = 0$ is in q. By the definition of Morley rank, (here we use the fact that Morley rank is equal to Cantor-Bendixson rank in ω-saturated structures, (see Lemma 5.6.3 in [1])) there exists $r \neq q$ in $S_n(\mathcal{K})$ such that the formula $\bigwedge_{i=1}^{k} f_i(\bar{t}) = 0$ is in r and $\mathrm{RM}(r) \geq \alpha$. So we have $\mathcal{I}_q \subsetneq \mathcal{I}_r$. By the definition of δ-dimension and the induction hypothesis we have $\dim_\delta \mathcal{I}_q > \dim_\delta \mathcal{I}_r \geq \alpha$. □

Using the correspondence between prime δ-ideals of $\mathcal{K}\{t_1, \ldots, t_n\}$ which contain $\mathcal{I}_{\tilde{p}}$ and prime δ-ideals of $\mathcal{K}\{t_1, \ldots, t_n\}/\mathcal{I}_{\tilde{p}} \cong \mathcal{K}\{\bar{a}\}$, we can rewrite the conclusion of Lemma 2.2 as $\mathrm{RM}(p) \leq \dim_\delta \mathcal{K}\{\bar{a}\}$ where $\bar{a} \in \mathcal{U}^n$ is a realization of $\tilde{p}$.

The following results are inspired by the works of J.Johnson in [5] and [4]. He defined in [5] the Krull dimension of a δ-ring in a general setting. Since we are dealing with ordinary differential rings only (i.e. differential rings with only one derivation), no such level of generality is required. Nevertheless we encourage readers who are interested in the results in several derivations to consult the original papers.

Let $\mathcal{K}$ be a δ-field (not necessary δ-closed) and $\mathcal{R}$ be a δ-integral domain which is also a finitely generated δ-$\mathcal{K}$-algebra. By the differential basis theorem, $\mathrm{Spec}_\delta \mathcal{R}$ is a Noetherian topological space. We fix a set of generators $a_1, \ldots, a_n$ for $\mathcal{R}$; let $R^{[-1]} = \mathcal{K}$ and

$$R^{[r]} = \mathcal{K}[\delta^j a_i : 1 \leq i \leq n, 0 \leq j \leq r]$$

for $r \geq 0$.

Lemma 2.3. *Let $\mathcal{L}/\mathcal{K}$ be a differential field extension. Let S be a subset of $\mathcal{L}$. If $b_1, \ldots, b_k$ in $\mathcal{L}$ are algebraically dependent over $\mathcal{K}(S)$, then $\delta b_1, \ldots, \delta b_k$ are algebraically dependent over the field $\mathcal{K}(S \cup \delta S \cup \{b_1, \ldots, b_k\})$.*

Proof. Let $P(y_1, \ldots, y_k)$ be a nonzero polynomial over $\mathcal{K}(S)$ with minimal total degree such that $P(b_1, \ldots, b_k) = 0$. Apply δ to both sides of this equation and we get

$$P^\delta(\bar{b}) + \sum_{i=1}^{k} \frac{\partial P}{\partial y_i}(\bar{b}) \delta b_i = 0 \tag{1}$$

where P^δ is the polynomial obtained by applying δ to the coefficients of P. Since $P \neq 0$, so $\frac{\partial P}{\partial y_i}$ is a nonzero polynomial and has total degree less that P for some $1 \leq i \leq k$. Therefore by our choice of P, $\frac{\partial P}{\partial y_i}(\bar{b}) \neq 0$. Thus (1) gives a nontrivial algebraic relation among the δb_i's over the field $\mathcal{K}(S \cup \delta S \cup \{b_1, \ldots, b_k\})$.

Note that for the case $k = 1$, we can conclude that δb_1 is even algebraic over $\mathcal{K}(S \cup \delta S)$. [1] This follows from the lemma and that b_1 itself is algebraic over $\mathcal{K}(S \cup \delta S)$. □

Let $L^{[r]}$ be the field of fractions of $R^{[r]}$ and $\mathcal{L}$ be the field of fractions of $\mathcal{R}$, note that $\mathcal{L}$ is a δ-field. For all sufficiently large r, the relation between r and the transcendence degree of $L^{[r]}$ over K is given by the following proposition.

Proposition 2.4. *There is a polynomial of the form $dt + b$, where d is the δ-transcendence degree of $\mathcal{L}/\mathcal{K}$ and b is a nonnegative integer, such that* $\operatorname{td}(L^{[r]}/\mathcal{K}) = dr + b$ *for all sufficiently large* $r \in \mathbb{N}$.

Proof. For any $r \geq 0$, we have $L^{[r]} = L^{[r-1]}(\delta^r a_1, \ldots, \delta^r a_n)$. First we want to show that $\operatorname{td}(L^{[r]}/L^{[r-1]})$ is a non-increasing function of r. If $L^{[r]}$ is algebraic over $L^{[r-1]} = 0$, then an application of Lemma 2.3 shows that $L^{[r+1]}$ is algebraic over $L^{[r]}$. So let us assume $\operatorname{td}(L^{[r]}/L^{[r-1]}) \geq 1$. Renumbering the generators if necessary, we may assume $\delta^r a_1, \ldots, \delta^r a_e$ forms a transcendence basis of $L^{[r]}/L^{[r-1]}$. Again by Lemma 2.3, $\delta^{r+1} a_i$ is algebraic over $L^{[r]}(\delta^{r+1} a_1, \ldots, \delta^{r+1} a_e)$ for every $i \geq e+1$. Since r is arbitrary, we see that $\operatorname{td}(L^{[r]}/L^{[r-1]})$ is a non-increasing function of r. Consequently there exist integers m and d such that $\operatorname{td}(L^{[r]}/L^{[r-1]}) = d \geq 0$, for all $r \geq m$. Also note that at each stage we add at least d to the transcendence degree; so $\operatorname{td}(L^{[m-1]}/\mathcal{K}) \geq md$. Thus for $r \geq m$, we have

$$\begin{aligned} \operatorname{td}(L^{[r]}/\mathcal{K}) &= \sum_{j=m}^{r} \operatorname{td}(L^{[j]}/L^{[j-1]}) + \operatorname{td}(L^{[m-1]}/\mathcal{K}) \\ &= (r - m + 1)d + \operatorname{td}(L^{[m-1]}/\mathcal{K}) = dr + b \end{aligned}$$

for some $b \geq d$. It remains to argue that d is the δ-transcendence degree of $\mathcal{L}/\mathcal{K}$. To prove this, pick a transcendence basis of $L^{[m]}/L^{[m-1]}$. Without loss of generality, say it is $\{\delta^m a_1, \ldots, \delta^m a_d\}$. If $d = 0$, then the required polynomial is simply a constant. On one hand, it follows from Lemma 2.3 that $\delta^j a_1, \ldots, \delta^j a_d$ are algebraically independent over $L^{[j-1]}$, for all $0 \leq j \leq m$. On the other hand, by (2.3) again, we conclude that $\delta^{m+1} a_i$ is algebraic over $L^{[m]}(\delta^{m+1} a_1, \ldots, \delta^{m+1} a_d)$, for each $i \geq d+1$. But $\operatorname{td}(L^{[m+1]}/L^{[m]}) = d$ hence $\delta^{m+1} a_1, \ldots, \delta^{m+1} a_d$ must remain algebraically independent over $L^{[m]}$.

[1] I would like to thank the referee for pointing out to me this argument and that the $k = 1$ case is enough to prove Proposition 2.4.

By repeating the same argument, we conclude that $\{a_1, \ldots, a_d\}$ is a maximal δ-independent subset of the generators so $\mathrm{td}_\delta(\mathcal{L}/\mathcal{K}) = d$.

Thus $dt + b$ is the polynomial with the required property. We call this the **Kolchin polynomial** [1] **of $\bar{a}$ over $\mathcal{K}$** (or of $\mathcal{R}$ if the generators of $\mathcal{R}$ over $\mathcal{K}$ are clear). $\square$

2.2 Relations between Ranks

For any $\mathcal{P} \in \mathrm{Spec}_\delta \mathcal{R}$, let $a_\mathcal{P} t + b_\mathcal{P}$ be the Kolchin polynomial of $\mathcal{R}/\mathcal{P}$. If $\mathcal{Q}$ is another prime δ-ideal containing $\mathcal{P}$, then for $r \in \mathbb{N}$ sufficiently large, $P^{[r]} := \mathcal{P} \cap R^{[r]} \subset Q^{[r]} := \mathcal{Q} \cap R^{[r]}$. Since $R^{[r]}$ is finitely generated as a $\mathcal{K}$-algebra, $\mathrm{td}((R^{[r]}/P^{[r]})/\mathcal{K}) > \mathrm{td}((R^{[r]}/Q^{[r]})/\mathcal{K})$. Note also that $(\mathcal{R}/\mathcal{P})^{[r]}$ and $R^{[r]}/P^{[r]}$ are isomorphic as rings. Thus $a_\mathcal{P} r + b_\mathcal{P} = \mathrm{td}((\mathcal{R}/\mathcal{P})^{[r]}/\mathcal{K}) > \mathrm{td}((\mathcal{R}/\mathcal{Q})^{[r]}/\mathcal{K}) = a_\mathcal{Q} r + b_\mathcal{Q}$, for all large enough r. That simply means $\omega a_\mathcal{P} + b_\mathcal{P} > \omega a_\mathcal{Q} + b_\mathcal{Q}$ as ordinals. Using this observation, we can show that the δ-dimension of a prime δ-ideal is bounded above by the value of its Kolchin polynomial at ω.

Proposition 2.5. *For any $\mathcal{P} \in \mathrm{Spec}_\delta \mathcal{R}, \dim_\delta \mathcal{P} \leq \omega a_\mathcal{P} + b_\mathcal{P}$.*

Proof. We prove this by induction on the δ-dimension of $\mathcal{P}$. Let $\dim_\delta \mathcal{P} = \alpha$. By definition of δ-dimension, for any $\beta < \alpha$ there exists $\mathcal{Q} \in \mathrm{Spec}_\delta \mathcal{R}$ such that $\mathcal{Q} \supset \mathcal{P}$ and $\beta \leq \dim_\delta \mathcal{Q} < \alpha$. By the induction assumption and the above discussion, we have $\dim_\delta \mathcal{Q} \leq \omega a_\mathcal{Q} + b_\mathcal{Q} < \omega a_\mathcal{P} + b_\mathcal{P}$. Since $\beta < \alpha$ is arbitrary, we conclude that $\omega a_\mathcal{P} + b_\mathcal{P} \geq \alpha = \dim_\delta \mathcal{P}$. $\square$

Corollary 2.6. $\dim_\delta \mathcal{R} < \omega(\mathrm{td}_\delta(\mathcal{R}/\mathcal{K}) + 1)$.

Proof. By Proposition 2.4, the Kolchin polynomial of the zero ideal is of the form $dt + b$ where $d = \mathrm{td}_\delta(\mathcal{R}/\mathcal{K})$. So by Proposition 2.5, we have $\dim_\delta \mathcal{R} \leq \omega d + b < \omega(d+1)$. $\square$

Now we can prove the result promised at the beginning of this section.

Theorem 2.7. *Let $p \in S_n(\mathcal{K})$. If $\mathrm{RM}(p)$ is a limit ordinal, then $\mathrm{RU}(p) = \mathrm{RM}(p)$.*

[1] It is also known as the dimension polynomial. In general, it is a polynomial with degree less than or equal to the number of derivations. For details, see [4], [6] or [10].

Proof. By taking a nonforking extension of p if necessary, we can assume $\mathcal{K}$ is ω-saturated. Let $(a_1, \ldots, a_n)$ be a realization of p. Without loss of generality, $a_1, \ldots, a_d$ forms a δ-transcendence base of $\mathcal{K}\langle a_1, \ldots, a_n\rangle$. By induction and Lascar inequalities, we have $\mathrm{RU}(p) \geq \mathrm{RU}(a_1, \ldots, a_d/\mathcal{K}) = \omega d$. On the other hand by Lemma 2.2 and Corollary 2.6, $\mathrm{RM}(p) \leq \dim_\delta \mathcal{K}\{\bar{a}\} < \omega(d+1)$. So all together, we have

$$\omega d \leq \mathrm{RU}(p) \leq \mathrm{RM}(p) < \omega(d+1).$$

Since the last inequality is strict, if $\mathrm{RM}(p)$ is a limit, it must be equal to ωd. This forces $\mathrm{RU}(p) = \mathrm{RM}(p)$. $\square$

The following is a general result about the various ranks that we have considered. The definition of RC is taken from [8].

Theorem 2.8. *Let T be a totally transcendental theory. If the following hold:*

1. *For any type p, if* $\mathrm{RM}(p)$ *is a limit ordinal, then* $\mathrm{RM}(p) = \mathrm{RU}(p)$
2. $\mathrm{RM} = \mathrm{RC}$

then $\mathrm{RM} = \mathrm{RU}$.

Proof. Let A be a subset of some model of T and p be an m type over A. We are going to show by induction on α that if $\mathrm{RM}(p) = \alpha$, then $\mathrm{RU}(p) = \alpha$.

When $\alpha = 0$, there is nothing to prove since p is algebraic. In this case, $\mathrm{RU}(p) = \mathrm{RM}(p) = 0$. Our assumption takes care of the case when α is a limit. Suppose $\mathrm{RM}(p) = \alpha+1$, RM and RU agree up to α. Since it is always the case that $\mathrm{RU}(p) \leq \mathrm{RM}(p)$, it suffices to show that $\mathrm{RU}(p) > \alpha$.

Pick $\varphi \in p$ such that $\mathrm{RM}(\varphi) = \mathrm{RM}(p) = \alpha + 1$ and φ isolates p among those types in $S_m(A)$ which have Morley rank greater than or equal to $\alpha+1$. By assumption (2), $\mathrm{RC}(\varphi) = \alpha + 1$ which means that there is a formula ψ with parameters in some $B \supseteq A$ such that $\mathrm{RC}(\psi) = \alpha$, ψ forks over A and $\forall \bar{x}[\psi(\bar{x}) \longrightarrow \varphi(\bar{x})]$. Choose q in $S_n(B)$ containing ψ such that $\mathrm{RC}(q) = \mathrm{RC}(\psi)$. Now by the induction hypothesis, $\mathrm{RM}(q) = \mathrm{RU}(q) = \alpha$. Since ψ forks over A, we have $\mathrm{RM}(q|_A) \geq \mathrm{RU}(q|_A) \geq \alpha + 1$. Moreover, we have $\varphi \in q|_A$. Therefore by our choice of φ, $q|_A = p$ so $\mathrm{RU}(p) \geq \alpha + 1$. $\square$

A direct consequence of (2.7) and (2.8) is

Corollary 2.9. *In* DCF_0*, if* $\mathrm{RM} = \mathrm{RC}$*, then* $\mathrm{RM} = \mathrm{RU}$*.*

By a result of Hrushovski and Scanlon [2], we know that RM and RU are in general not equal in DCF_0 (also see the addendum). Using this result and Corollary 2.9, we conclude that RM and RC are not always equal in DCF_0 as well.

Addendum

Recently Hrushovski and Scanlon have shown that in DCF_0 Morley rank and Lascar rank are different (see [2]). By Corollary 2.9, we know that $\mathrm{RM} \neq \mathrm{RC}$ in DCF_0 as well. I would like to thank Ehud Hrushovski for pointing out to me the following example which shows that Morley rank and Lascar rank of a type can be different even if the Lascar rank of the type is a limit ordinal. I also benefited from discussing this example with David Marker and Tom Scanlon.

Let a be δ-transcendental over $\mathbb{Q}$. Let E_a be the elliptic curve with j-invariant a. Pick b such that $q = \mathrm{tp}(b/a)$ is the generic type of the Manin kernel of E_a.

Claim 1 $\mathrm{RU}(a, b) = \omega$.

It suffices to show that if $\{a, b\}$ forks over some $\mathcal{L}$, then $\mathrm{RU}(a, b/\mathcal{L})$ is finite. Suppose $a \mathrel{\perp\!\!\!\smile} \mathcal{L}$, then $tp(\mathcal{L}/a)$ is a nonforking extension of $tp(\mathcal{L}/\emptyset)$. By Proposition 2.8 in [3], q is orthogonal to $tp(\mathcal{L}/\emptyset)$ and hence to $tp(\mathcal{L}/a)$. Since b realizes q, this means $b \mathrel{\perp\!\!\!\smile}_a \mathcal{L}$. Putting this together with $a \mathrel{\perp\!\!\!\smile} \mathcal{L}$, we have $ab \mathrel{\perp\!\!\!\smile} \mathcal{L}$ contradicting our assumption. Therefore $a \not\mathrel{\perp\!\!\!\smile} \mathcal{L}$ and hence $\mathrm{RU}(a/\mathcal{L})$ is finite. Moreover, by Proposition 2.6 in [3] q is strongly minimal hence $\mathrm{RU}(b/\mathcal{L}\langle a\rangle) \leq 1$. Now it follows from the Lascar inequalities that $\mathrm{RU}(a, b/\mathcal{L})$ is finite.

Claim 2 $\mathrm{RM}(a, b) = \omega + 1$.

Let $p(x, y)$ be a nonforking extension of $tp(a, b)$ over some ω-saturated differentially closed field $\mathcal{K}$. By strongly minimality of the Manin kernel, p is isolated by the following collection of formulas:

- x is δ-transcendental over $\mathcal{K}$.
- $\mu(y) = 0$ where μ is the Manin map of E_x.
- y is not algebraic over $\mathcal{K}\langle x\rangle$.

Let Φ be any finite sub-collection of the above formulas containing "$\mu(y) = 0$". Suppose (c, d) is a realization of Φ. Since $\mu(d) = 0$, d is δ-algebraic over c. So if $\mathrm{RM}(c, d) > \omega$, c has to be δ-transcendental over $\mathcal{K}$. By rank considerations, d cannot be algebraic over c. So actually we have $p = tp(c, d/\mathcal{K})$. This shows that $\mathrm{RM}(p) \leq \omega + 1$. On the other hand there are only finitely many inequations in Φ, so Φ can be satisfied by (c, d) where c is δ-transcendental over $\mathcal{K}$ and d is a torsion point of E_c. Such a tuple will have Morley rank ω. Thus p is a limit point of Morley rank ω types. Therefore $\mathrm{RM}(a, b) = \mathrm{RM}(p) = \omega + 1$.

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