Length Spectra of Natural Numbers

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Abstract

Two numbers are spectral equivalent if they have the same length spectrum. We show how to compute the equivalence classes of this relation. Moreover, we show that these classes can only have either $1, 2$ or infinitely many elements.

1 Introduction

Some numbers can be written as a sum of consecutive integers, for example, $9 = 2 + 3 + 4$ but some cannot, for example, 8. The following is a beautiful characterization of this phenomenon:

Theorem 1.1. *A number is a sum of consecutive integers if and only if it is not a power of 2.*

Proofs of Theorem 1.1 can be found in [1, 2, 3].

To simplify the subsequent discussion, let us call a sequence of consecutive natural numbers a **decomposition of n** if its terms sum to n. The **length** of a decomposition is the number of terms in the decomposition and the **parity** of a decomposition is the parity of its length. A **trivial** decomposition is a decomposition of length 1. Clearly every number n has a trivial decomposition, namely (n). The following result in [3] (also in [1, 2]) is fundamental and will be used frequently throughout this article.

Theorem 1.2. *Let n be a natural number and k be an odd factor of n. If $k^2 < 2n$, then the sequence*

$$\frac{n}{k} - \frac{k-1}{2}, \quad \frac{n}{k} - \frac{k-1}{2} + 1, \quad \cdots, \quad \frac{n}{k} + \frac{k-1}{2} \tag{1}$$

is an odd decomposition of n of length k. On the other hand, if $k^2 > 2n$, then the sequence

$$\frac{k-1}{2} - \frac{n}{k} + 1, \quad \frac{k-1}{2} - \frac{n}{k} + 2, \quad \cdots, \quad \frac{n}{k} + \frac{k-1}{2} \tag{2}$$

is an even decomposition of n of length $2n/k$. Moreover, every decomposition of n has one of these forms.

Theorem 1.2 explicitly demonstrates a 1-to-1 correspondence between the odd factors of n and its decompositions. Since the powers of 2 are the only numbers having no odd factors other than 1, their decompositions can only be trivial. This establishes Theorem 1.1 as a consequence of Theorem 1.2.

The **length spectrum** (or simply the **spectrum**) of a number n, denoted by $\text{lspec}(n)$, is the set of lengths of the decompositions of n. According to Theorem 1.2, the spectrum of n is the set

$$\{k\colon k \text{ odd},\ k \mid n,\ k^2 < 2n\} \cup \{2n/k\colon k \text{ odd},\ k \mid n,\ k^2 > 2n\}. \tag{3}$$

As an example, we list in Table 1 the decompositions of the number 45 along with their lengths, parities and associated odd factors.

factor	decomposition	length	parity
1	(45)	1	odd
45	(22, 23)	2	even
3	(14, 15, 16)	3	odd
5	(7, 8, 9, 10, 11)	5	odd
15	(5, 6, 7, 8, 9, 10)	6	even
9	(1, 2, 3, 4, 5, 6, 7, 8, 9)	9	odd

Table 1: The spectrum of 45 is $\{1, 2, 3, 5, 6, 9\}$

We say that two numbers are **spectral equivalent** if they have the same length spectrum. The **spectral class** of n, denoted by $L(n)$, is the equivalence class of n under spectral equivalence. With these notions, Theorem 1.1

can be restated in the following way: The powers of 2 form a spectral class with $\{1\}$ as their common spectrum. The odd primes form another interesting spectral class. According to Theorem 1.2, they are the numbers having $\{1, 2\}$ as spectrum. These two examples motivate the following question:

Given n, can we compute its spectral class?

We will give an algorithm in Section 5 which answers this question in an affirmative way. We also derive from it another algorithm which solve the following problem.

Given a finite set of numbers S, compute the set of numbers with S as their common spectrum.

2 Properties of Length Spectra

The following notations and conventions will be adopted throughout the rest of this article.

- For a set A, we write $|A|$ for its cardinality.
- For a set of natural numbers A, we write A_0 and A_1 for the set of even and odd elements of A, respectively.
- For a rational number c and a set of natural numbers A, we write cA for the set $\{ca \colon a \in A\}$.
- For a prime p and a natural number m, we write $v_p(m)$ for the exponent of p in the prime factorization of m.

Let n be a fixed but arbitrary natural number, we use $(k_i)_{i=1}^s$ to denote the list of odd factors of n in ascending order. So $k_1 = 1$ and $n = 2^\nu k_s$ where $\nu = v_2(n)$. We use r to denote the largest index such that $k_r^2 < 2n$. Let us note that for $1 \le i \le s$, k_j and k_{s-j+1} are complementary factors of k_s, i.e. $k_j k_{s-j+1} = k_s$ hence the length spectrum of n can be re-written as

$$\{k_i \colon 1 \le i \le r\} \cup 2^{\nu+1}\{k_i \colon 1 \le i \le s - r\}. \tag{4}$$

The first important observation about length spectra is:

Theorem 2.1. *The number of even decompositions of a number is at most the number of its odd decompositions.*

Proof. Suppose k_j corresponds to an even decomposition of n. By Theorem 1.2, $k_j^2 > 2n$ and so $k_{s-j+1}^2 = k_s^2/k_j^2 < (2n)^2/2n = 2n$. Therefore, again by Theorem 1.2, k_{s-j+1} corresponds to an odd decomposition of n. The theorem now follows since the map $k \mapsto k_s/k$ is 1-to-1. □

In the light of Theorem 2.1, we call a spectrum

- **balanced** if it has an equal number of even and odd elements;
- **unmixed** if it has no even elements;
- **lopsided** if it is neither balanced nor unmixed.

Our next result characterizes the numbers with an unmixed spectrum.

Theorem 2.2. *The set of numbers with an unmixed spectrum is*

$$\{2^\alpha k \colon \alpha \geq 0,\ k \textit{ odd},\ 2^{\alpha+1} > k\}.$$

Proof. It follows from (3) that a number with no even decompositions if and only if it is of the form $2^\alpha k$ with $\alpha \geq 0$, k odd and $2^{\alpha+1}k > k^2$, i.e. $2^{\alpha+1} > k$. □

Numbers with a balanced spectrum are trickier to capture. For a natural number k, let $q(k)$ be the minimum of the ratios m'/m where $m \leq m'$ are complementary factors of k. Note that by definition $q(k) \geq 1$. Moreover, for a pair m, m' of complementary factors of k, $m'/m = q(k)$ if and only if no factor of k is strictly in between m and m'. Also, it may be worth pointing out that $q(k)$ measures how far is k from being a prefect square: k is a prefect square if and only if $q(k) = 1$.

Theorem 2.3. *The set of numbers with a balanced spectrum is*

$$\{2^\alpha k \colon \alpha \geq 0,\ k \textit{ odd},\ q(k) > 2^{\alpha+1}\}.$$

Proof. Suppose $2^\alpha k$ is a member of the set in display. Let m, m' be a pair of complementary factors of k with $m'/m = q(k)$. Then $m'/m > 2^{\alpha+1}$, hence $m' > 2^{\alpha+1}m$ and so

$$m'^2 > 2^{\alpha+1}mm' = 2^{\alpha+1}k > m^2. \tag{5}$$

Since no factor of k is strictly in between m and m', the inequalities in (5) imply the odd (resp. even) decompositions of $2^\alpha k$ correspond precisely to those factors of k that are $\leq m$ (resp. $\geq m'$). Thus the assignment $l \mapsto mm'/l$ induces an injective map from the odd decompositions of $2^\alpha k = 2^\alpha mm'$ to its even decompositions. Therefore, by Theorem 2.1, $2^\alpha k$ has a balanced spectrum.

Conversely, suppose n has a balanced spectrum. Then we have (in the notations introduced earlier) $s = 2r$. Hence $n = 2^\nu k_r k_{r+1}$ and $k_{r+1}/k_r = q(k_r k_{r+1})$. Moreover, $k_{r+1}^2 > 2n = 2^{\nu+1} k_r k_{r+1}$, therefore $k_{r+1}/k_r > 2^{\nu+1}$ and so n belongs to the set displayed in the statement of the theorem. □

3 Spectral Classes

In this section, we determine the spectral class of a number according to the type of its spectrum. We begin with an observation which is clear from the form of the spectrum given in (4):

Proposition 3.1. *If n has an even decomposition, then the highest power of 2 dividing n is half the least even element of $\operatorname{lspec}(n)$. In fact, $v_2(n) = v_2(e) - 1$ for any even element e of $\operatorname{lspec}(n)$.*

Let S be a finite set of natural numbers. Recall that S_i is the set of elements of S congruent to $i \pmod 2$. We define $D(S)$, the **difference set** of S, to be S_1 with its least $|S_0|$ elements removed if $|S_0| \leq |S_1|$, or the empty set otherwise.

Proposition 3.2. *If a number has more odd than even decompositions, then its greatest odd factor is the product of the maximum and minimum of the difference set of its spectrum.*

Proof. Suppose $S = \operatorname{lspec}(n)$ and $|S_1| > |S_0|$. In this case, $D(S)$ is the non-empty set $S_1 \setminus 2^{-(\nu+1)} S_0 = \{k_{s-r+1}, \ldots, k_r\}$. Since $k_{s-r+1} k_r = k_s$, the largest odd factor of n, the proposition follows. □

Theorem 3.3. *If lspec(n) is lopsided, then $L(n) = \{n\}$.*

Proof. If $\operatorname{lspec}(n)$ is lopsided, then by Proposition 3.1 and 3.2 both $v_2(n)$ and k_s, and therefore n, can be recovered from $\operatorname{lspec}(n)$. □

To illustrate the results that we have just proved, let us decide whether the set $S := \{1, 3, 4, 5, 9, 12\} = \{1, 3, 5, 9\} \cup \{4, 12\}$ is a spectrum. First, let us note that $D(S) = \{5, 9\}$. So if S is a spectrum, say $S = \operatorname{lspec}(n)$, then by Proposition 3.2 the largest odd factor of n is $5 \cdot 9 = 45$. By Proposition 3.1, $v_2(n) = v_2(4) - 1 = 1$. Therefore, n can only be 90. One then finds that S is indeed a spectrum by verifying $\operatorname{lspec}(90) = S$.

Next we determine the spectral class of n when its spectrum is unmixed.

Theorem 3.4. *If* $\operatorname{lspec}(n)$ *is unmixed, then*

$$L(n) = \{2^\alpha m \colon \alpha \geq 0,\ 2^{\alpha+1} > m\}$$

where m *is the largest odd element of* $\operatorname{lspec}(n)$*.*

Proof. Suppose $\operatorname{lspec}(n)$ is unmixed. Let m be the largest odd element of $\operatorname{lspec}(n)$, S be the set of factors of m and R be the set on the right-hand-side of the equation. First, since $\operatorname{lspec}(n)$ is unmixed, it follows easily from (4) that $\operatorname{lspec}(n) = S$. By Theorem 1.2, every element of R has spectrum S. By Theorem 2.2, the converse is true hence R is precisely the set of number with spectrum S and the theorem follows. ☐

For a finite set of natural numbers S, we define the **exceptional set** of S to be the set

$$E(S) = \{a \in S_1^2 \colon a > m_0,\ F_{<a}(am_1) = S_1\}$$

where S_1^2 is the set $\{bc \colon b, c \in S_1\}$, $m_i = \max S_i$ $(i = 0, 1)$ and $F_{<k}(l)$ denotes the set of factors of l which are strictly less than k. The following simple facts about elements of exceptional sets will come in handy for our subsequent arguments:

Lemma 3.5. *Suppose* S *is a balanced spectrum and* $a \in E(S)$*, then*

(i) a *has a proper prime factor; in particular,* a *is not a prime.*

(ii) every proper factor of a *is in* S_1*.*

(iii) $a/m_1 = q(m_1 a)$*.*

Proof. By definition, $a > m_0 \geq 2m_1 \geq 2$. Moreover, $a \in S_1^2 \setminus S_1$ therefore a cannot be a prime and so (i) follows. Every proper factor of a is clearly a member of $F_{<a}(m_1 a)$ which is S_1 and so (ii) follows. To show (iii), it suffices to show that no factor of $m_1 a$ is strictly in between m_1 and a. If not, then $F_{<a}(m_1 a)$, i.e. S_1 will contain a number larger than m_1, a contradiction. ☐

The next lemma is crucial to our analysis of balanced spectra.

Lemma 3.6. *Suppose* $S = \operatorname{lspec}(n)$ *is balanced, then*

$$\frac{m_0}{2}E(S) \subseteq L(n) \subseteq \frac{m_0}{2}(P(S) \cup E(S)).$$

where $P(S)$ *is the set of primes that are larger than* m_0.

Proof. Since $S = \operatorname{lspec}(n)$ is balanced, it is of the form $S_1 \cup 2^{\nu+1}S_1$ where $\nu = v_2(n)$. In particular, $m_0 = 2^{\nu+1}m_1$.

Pick $a \in E(S)$, and let n' be $m_0a/2 = 2^\nu m_1 a$. By Lemma 3.5 (iii), $a/m_1 = q(m_1a)$ and so the inequality $a > m_0 = 2^{\nu+1}m_1$ implies $q(m_1a) > 2^{\nu+1}$. Therefore, $\operatorname{lspec}(n')$ is balanced according to Theorem 2.3. Note that the inequality $a > m_0$ also implies

$$a^2 > m_0a = 2n' = 2^{\nu+1}m_1a > m_1^2.$$

These inequalities together with Theorem 1.2 and the fact that $a/m_1 = q(m_1a)$ imply the set of odd elements of $\operatorname{lspec}(n')$ is $F_{<a}(m_1a)$. But $F_{<a}(m_1a)$ is simply S_1, as a is a member of $E(S)$. Therefore,

$$\operatorname{lspec}(n') = \operatorname{lspec}(n')_1 \cup 2^{v_2(n')+1}\operatorname{lspec}(n')_1 = S_1 \cup 2^{\nu+1}S_1 = \operatorname{lspec}(n).$$

So we conclude that $n' \in L(n)$.

To show the second inclusion, suppose $n' \in L(n)$, i.e. $\operatorname{lspec}(n') = S_1 \cup 2^{\nu+1}S_1$. Thus m_1 is the largest odd factor of n' corresponding to an odd decomposition. Since $\operatorname{lspec}(n')$ is balanced, the largest odd factor of n' is of the form m_1a for some odd number $a > m_1$ such that $a/m_1 = q(m_1a)$. By Proposition 3.1, $v_2(n') = \nu$ and so $n' = 2^\nu m_1 a = m_0a/2$. By Theorem 2.3, $q(m_1a) > 2^{\nu+1}$. Therefore, $a > 2^{\nu+1}m_1 = m_0$. Since no factor of m_1a is strictly in between a and m_1, $F_{<a}(m_1a)$ is the set of odd factors of n' not exceeding m_1 and that is S_1. Moreover, since every proper factor of a belongs to $F_{<a}(m_1a) = S_1$, that means either $a \in S_1^2$ or a is a prime. In the former case, $a \in E(S)$ and in the latter case, $a \in P(S)$. □

Before stating our next result, which gives the spectral class of n when its spectrum is balanced, we need to introduce one more concept. Suppose S is a spectrum. By Theorem 1.2, S_1 contains the factors of its elements. In particular, the sets of factors of m_1 is always a subset of S_1. In the light of this observation, we call a balanced spectrum S **non-excessive** if S_1 is precisely the set of factors of m_1; otherwise we call S **excessive**. Note that we use the word (non-)excessive to describe balanced spectra only.

Theorem 3.7. *Suppose $S = \operatorname{lspec}(n)$ is balanced. Then*

(i) $L(n) = \dfrac{m_0}{2}(P(S) \cup E(S))$, *if S is non-excessive; or*

(ii) $L(n) = \dfrac{m_0}{2}E(S)$, *if S is excessive.*

Proof. To proof (i), thanks to Lemma 3.6, we only need to show that the set $m_0P(S)/2$ is a subset of $L(n)$. Take $p \in P(S)$ and let $n' = m_0p/2 = 2^\nu m_1 p$ where $\nu = v_2(n)$. Since p is prime, the odd factors of n' are the factors of m_1 and their p multiples. The inequalities $p > m_0 = 2^{\nu+1}m_1 > m_1$ ensure

$$p^2 > m_0 p = 2n' > m_1^2.$$

Therefore, the factors of m_1 correspond to the odd decompositions of n' while their p multiplies correspond to the even decompositions of n' (Theorem 1.2). Thus, $\operatorname{lspec}(n')$ is balanced and the set of odd elements of $\operatorname{lspec}(n')$ coincides with the set of factors of m_1. Since S is non-excessive, the set of factors of m_1 equals S_1. So we actually have

$$\operatorname{lspec}(n') = \operatorname{lspec}(n')_1 \cup 2^{v_2(n')+1} \operatorname{lspec}(n')_1 = S_1 \cup 2^{\nu+1} S_1 = \operatorname{lspec}(n).$$

This finishes the proof of Part (i). Incidentally, the argument above also shows that if $S = \operatorname{lspec}(n)$ is balanced then every element of $m_0P(S)/2$ has non-excessive spectrum. Consequently, $L(n)$ and $m_0P(S)/2$ do not intersect if S is excessive. Thus Part (ii) follows from Lemma 3.6 as well. ☐

4 Structures of Exceptional Sets

We study of the structures of exceptional sets in this section. As a result, we prove a rather curious fact: a spectral class can only have either $1, 2$ or infinitely many elements. We start by making the following conventions and definitions. Throughout this section, S denotes a balanced spectrum and m denotes the largest odd element of S, moreover:

- For a prime p, let γ_p denote the largest integer such that $p^{\gamma_p} \in S_1$ and we write μ_p for $v_p(m)$. The **excessive index of p with respect to S** is defined to be $\epsilon_p := \gamma_p - \mu_p$. Note that ϵ_p is always non-negative.

- A prime p is called an **excessive prime** of S if $\epsilon_p > 0$; otherwise p is called a **non-excessive prime** of S. We also say that p is **excessive (non-excessive) with respect to** S if it is an excessive (a non-excessive) prime of S. Note that 2 is always a non-excessive prime of S. Also, every excessive prime of S is in S_1 and every non-excessive prime of S that is in S_1 divides m.

- The **excessive number** of S is defined to be the product $e_S := \prod p^{\epsilon_p}$ where p runs through the primes. Note that S is excessive if and only if S has an excessive prime if and only if $e_S > 1$.

- Every $a \in E(S)$ can be written as $e_a n_a$ where e_a (n_a) is a product of (non-)excessive primes. The numbers e_a and n_a are called the **excessive part** and the **non-excessive part** of a, respectively. Note that either e_a or n_a can be 1 but not both since $a > 1$.

The next two lemmas tell us what kind of factors that an element of $E(S)$ can/must have.

Lemma 4.1. *For every prime p, p^{ϵ_p} divides every element of $E(S)$.*

Proof. For any $a \in E(S)$, since $p^{\gamma_p} \in S_1$, $p^{\gamma_p} \mid ma$ and so $p^{\epsilon_p} = p^{\gamma_p - \mu_p} \mid a$. ☐

Lemma 4.2. *If q is a non-excessive prime of S dividing some element of $E(S)$ then $q > p^{\mu_p}$ for any prime p other than q.*

Proof. Suppose q is non-excessive and $q \mid a$ for some $a \in E(S)$. Then q^{μ_q+1} divides ma; moreover for any prime $p \neq q$, if $q < p^{\mu_p}$ then

$$q^{\mu_q+1} \leq \frac{mq}{p^{\mu_p}} < m < a.$$

But that means $q^{\mu_q+1} \in F_{<a}(ma) = S_1$, contradicting the fact that q is non-excessive. Therefore, we must have $p^{\mu_p} < q$. ☐

Proposition 4.3. *For every $a \in E(S)$,*

(i) e_a *is divisible by* e_S.

(ii) n_a *is either* 1 *or a power of the largest non-excessive prime of* S *in* S_1. *In particular, if* $n_a > 1$ *then* S_1 *contains a non-excessive prime of* S.

(iii) *if* $n_a > 1$, *then* $e_a = e_S$.

Proof. By Lemma 4.1, $e_S \mid a$. Since e_S is a product of excessive primes, so in fact, e_S divides e_a.

By Lemma 3.5 (i) and (ii), every prime factor of a is in S_1. So n_a is a product of non-excessive primes (of S) in S_1. Therefore, $n_a = 1$ if S_1 contains no non-excessive primes of S. So suppose otherwise and let q be the largest non-excessive prime of S in S_1. If q_0 is another non-excessive prime of S dividing n_a, then $q_0 \in S_1$ and so $q_0 < q$. However, by Lemma 4.2 $q^{\mu_q} < q_0$ and this leads to a contradiction since $\mu_q \geq 1$. So we conclude that n_a can have no prime factors other than q, therefore Part (ii) follows.

Suppose $n_a > 1$ then, by Part (ii), a is of the form $e_a q^\beta$ where q is the largest non-excessive prime of S in S_1 and $\beta \geq 1$. For a prime p, let us write α_p for $v_p(a)$. Consider the factor $p^{\mu_p+\alpha_p}$ of ma. For $p \neq q$, $p^{\alpha_p} \mid e_a$ and by Lemma 4.2 $p^{\mu_p} < q$. Therefore, $p^{\mu_p+\alpha_p} < a$ and so $p^{\mu_p+\alpha_p} \leq m$ since $a \in E(S)$. Consequently, $\mu_p + \alpha_p \leq \gamma_p$, i.e. $\alpha_p \leq \gamma_p - \mu_p = \epsilon_p$. But by Part (i), $\epsilon_p \leq \alpha_p$. Therefore we conclude that $\alpha_p = \epsilon_p$ for every prime $p \neq q$ and hence $e_a = e_S$. ☐

We should point out that it is possible for a balanced spectrum (other than $\{1, 2\}$) to have no non-excessive odd primes (see Example 5.8). Also, n_a in the above proposition may still be 1 even S_1 contains a non-excessive prime of S.

Next we give a characterization of non-excessive spectra in terms of exceptional sets.

Theorem 4.4. *A balanced spectrum S is non-excessive if and only if $E(S) = \emptyset$ or $E(S) = \{q^{\mu_q+1}\}$ for some prime q. In particular, the size of the exceptional set of a non-excessive spectrum is at most one.*

Proof. Suppose S is non-excessive and $a \in E(S)$. Since every prime is non-excessive with respect to S, $e_a = 1$ and so $n_a > 1$. Therefore, by Proposition 4.3 (ii), a is a power of the largest non-excessive prime q in S_1 (since S is non-excessive, so in fact q is outright the largest prime in S_1). Since $a > m \geq q^{\mu_q}$, so on one hand $a \geq q^{\mu_q+1}$; on the other hand $q^{\mu_q+1} \mid ma$ but $q^{\mu_q+1} \notin S_1$ thus $a \leq q^{\mu_q+1}$. Therefore, $E(S)$ must be the singleton $\{q^{\mu_q+1}\}$ if it is non-empty.

To show the other implication, let us note that if $E(S)$ is empty, then S is non-excessive by Theorem 3.7 (ii). So let us assume $E(S) = \{q^{\mu_q+1}\}$ for some prime q. Then every $k \in S_1$ divides mq^{μ_q+1} and $k < q^{\mu_q+1}$. Thus

$v_q(k) \le \mu_q$. Moreover, for any prime l other than q, $v_l(k) \le v_l(mq^{\mu_q+1}) = v_l(m)$. Therefore, we conclude that $k \mid m$ and hence S is non-excessive. □

The next result was a surprise to us.

Theorem 4.5. *$|E(S)| \le 2$.*

Proof. Suppose $|E(S)| > 1$, then S is excessive according to Theorem 4.4. Let p be an excessive prime of S. By Lemma 4.1, p^{ϵ_p} divides every element of $E(S)$. Moreover, since $\epsilon_p \ge 1$, by Lemma 3.5 (ii) $p^{-1}E(S)$ and hence $p^{-\epsilon_p}E(S)$ is a subset of S_1.

Let (u_i) be the list of elements of $U := p^{-\epsilon_p}E(S)$ in ascending order. For $i \ge 2$, since $m < p^{\epsilon_p}u_1 < p^{\epsilon_p}u_i \in E(S)$, therefore $p^{\epsilon_p}u_1$ must not divide $mp^{\epsilon_p}u_i$. In other words, u_1 does not divide mu_i. However, since $p^{\epsilon_p-1}u_1 \in p^{-1}E(S) \subseteq S_1$, $p^{\epsilon_p-1}u_1 \mid mp^{\epsilon_p}u_i$, i.e. $u_1 \mid mpu_i$. That means $v_p(u_1) \le v_p(mu_i)+1$ and for any prime l other than p, $v_l(u_1) \le v_l(mpu_i) = v_l(mu_i)$. So the fact that u_1 does not divide mu_i implies $v_p(u_1) > v_p(mu_i)$. Therefore, we must have

$$v_p(u_1) = v_p(mu_i) + 1 = \mu_p + v_p(u_i) + 1.$$

We claim that for $i \ge 2$, u_i is not divisible by p. If not, then $v_p(u_1) > \mu_p + 1$ and so p^{γ_p+1} would be a proper factor of $p^{\epsilon_p}u_1$ and hence, by Lemma 3.5 (ii), an element of S_1, a contradiction. Therefore, we conclude that

$$E(S) = p^{\epsilon_p}\{p^{\mu_p+1}v_1, u_2, \ldots, u_t\} = \{p^{\gamma_p+1}v_1, p^{\epsilon_p}u_2, \ldots, p^{\epsilon_p}u_t\}$$

for some $v_1, u_2, \ldots, u_t$ not divisible by p. Since $p^{\gamma_p+1} \notin S$, then again by Lemma 3.5 (ii) $v_1 = 1$ and so p^{γ_p+1} is the least element of $E(S)$.

As elements of S_1 the u_i's all divide mp^{γ_p+1}. But for $i \ge 2$, u_i and p are relatively prime so u_i must divide m. Therefore, if $|E(S)| > 2$, then we would have $p^{\epsilon_p}u_2 \mid mp^{\epsilon_p}u_3$ and $m < p^{\epsilon_p}u_2 < p^{\epsilon_p}u_3$, contradicting $p^{\epsilon_p}u_3 \in E(S)$. So we conclude that $|E(S)| \le 2$. □

Theorem 4.6. *Every spectral class has either* $1, 2$ *or infinitely many elements. Moreover, for any* n*,*

- $|L(n)| = 1$ *if and only if* $\operatorname{lspec}(n)$ *is lopsided or excessive with an exceptional set of size* 1.
- $|L(n)| = 2$ *if and only if* $\operatorname{lspec}(n)$ *is excessive with an exceptional set of size* 2.

- *$L(n)$ is infinite if and only if* lspec(n) *is unmixed or non-excessive.*

Proof. Suppose a spectral class $L(n)$ is finite, then $S :=$ lspec(n) must be either lopsided or excessive (Theorem 3.4 and 3.7 (i)). In the former case, $|L(n)| = |\{n\}| = 1$ (Theorem 3.3); in the latter case, $1 \leq |L(n)| = |E(S)| \leq 2$ according to Theorem 3.7 (ii) and Theorem 4.5. So we establish the first statement. The three equivalences are simply reorganizing what we have already proved in Theorem 3.3, 3.4 and 3.7. $\square$

We conclude this section with a few more precise descriptions of the exceptional sets.

Proposition 4.7. *If $e_S > m$, then $E(S) = \{e_S\}$.*

Proof. If $e_S > m$, then in particular e_S is greater than 1 so S is excessive and hence $|E(S)| \geq 1$. Let $a \in E(S)$, by Lemma 4.1, $e_S \mid a$. However, since $e_S > m$, therefore by Lemma 3.5 (ii), $e_S = a$. $\square$

Proposition 4.8. *Suppose $|E(S)| = 1$ then the unique element of $E(S)$ is either a product of excessive primes or of the form $e_S q^\beta$ ($\beta \geq 1$) where q is the largest non-excessive primes of S in S_1.*

Proof. Let a be the unique element of $E(S)$. If a is not a product of excessive primes then by Proposition 4.3 (ii) and (iii), a is of the form $e_S q^\beta$ with $\beta \geq 1$. $\square$

Proposition 4.9. *Suppose $|E(S)| > 1$ then S has a unique excessive prime p and $E(S)$ is of the form $\{p^{\gamma_p+1}, p^{\epsilon_p} q^\beta\}$ ($\beta \geq 1$) where p and q are the two largest primes in S_1. Moreover,*

(i) if p is the largest prime in S_1, then $\epsilon_p = \gamma_p$.

(ii) if p is not the largest prime in S_1, then $\beta = 1$.

Proof. We have already proved (Theorem 4.5) that $|E(S)| > 1$ implies $E(S)$ is of the form $\{p^{\gamma_p+1}, p^{\epsilon_p} u\}$ where p is an excessive prime of S, $u > p^{\mu_p+1}$ and u is not divisible by p. By Lemma 4.1, every excessive prime of S divides p^{γ_p+1}, therefore p is the only excessive prime of S and so u is the non-excessive part of $p^{\epsilon_p} u$. Since $u > 1$, by Proposition 4.3 u is a positive power of q where q is the largest non-excessive prime of S in S_1. Therefore, we conclude that $E(S)$ must be of the form $\{p^{\gamma_p+1}, p^{\epsilon_p} q^\beta\}$ for some $\beta \geq 1$.

Since q is the largest non-excessive prime in S_1 and p is the unique excessive prime in S_1, therefore if p is the largest prime in S_1 then q must be the second largest prime in S_1. Since q divides $p^{\epsilon_p} q^{\beta} \in E(S)$, by Lemma 4.2, $q > p^{\mu_p}$. Therefore, μ_p must be 0, i.e. $\epsilon_p = \gamma_p$. This completes the proof of (i).

Suppose p is not the largest prime in S_1, then the largest prime in S_1 is non-excessive and so must be q. We claim that in this case β is actually 1. First, note that $m < p^{\gamma_p+1} < p^{\gamma_p} q$ and since $p^{\gamma_p} q$ divides $mp^{\epsilon_p} q^{\beta}$, $p^{\epsilon_p} q^{\beta} \leq p^{\gamma_p} q$ i.e. $q^{\beta-1} \leq p^{\mu_p}$. But by Lemma 4.2, we also have $p^{\mu_p} < q$. Therefore, β must be 1.

To finish the proof of (ii), we argue that p must be the second largest prime in S_1. Since $q^{\mu_q+1} \mid mp^{\epsilon_p} q$ and q is non-excessive, $q^{\mu_q+1} \geq p^{\epsilon_p} q$. In other words, $p^{\mu_p} q^{\mu_q} \geq p^{\gamma_p}$. So if there were a prime $l \in S_1$ strictly in between p and q then $p^{\mu_p} l q^{\mu_q} > p^{\gamma_p+1} > m$. But since l is non-excessive, it divides m and therefore $p^{\mu_p} l q^{\mu_q} \mid m$, a contradiction. □

5 Examples and Algorithms

We give some examples here to illustrate the results in previous sections.

Example 5.1. The smallest number with an unmixed spectrum is 1. Since the spectrum of 1 is $\{1\}$, it follows from Theorem 3.4 that $L(1)$ is the set of powers of 2.

Example 5.2. The smallest number with a balanced spectrum is 3. The spectrum of 3 is $\{1, 2\}$ which is non-excessive and has an empty exceptional set. By Theorem 3.7 (i), $L(3)$ is the set of odd primes.

Example 5.3. The smallest number with a lopsided spectrum is 9. The spectrum of 9 is $\{1, 2, 3\}$. By Theorem 3.3, $L(9) = \{9\}$.

Example 5.4. The number 21 is the smallest number with a spectrum that has a non-empty exceptional set. The spectrum of 21 is $\{1, 2, 3, 6\}$. It is non-excessive with exceptional set $\{9\}$. By Theorem 3.7 (i),

$$L(21) = \{27\} \cup \{3p : p \text{ prime} > 6\}.$$

Example 5.5. The smallest number with an excessive spectrum is 75. The exceptional set of $\mathrm{lspec}(75) = \{1, 2, 3, 5, 6, 10\}$ is $\{15\}$. By Theorem 3.7 (ii), $L(75) = \{75\}$.

Example 5.6. The smallest number with two elements in the exceptional set of its spectrum is 175. The spectrum of 175 is $\{1, 2, 5, 7, 10, 14\}$. The exceptional set of lspec(175) is $\{25, 35\}$ (c.f. Proposition 4.9 (ii)). By Theorem 3.7 (ii),

$$L(175) = \frac{14}{2}\{25, 35\} = \{175, 245\}.$$

Example 5.7. The proof of Theorem 4.5 would be considerably simpler if every excessive spectrum contains an odd prime not dividing its largest odd element. However, it is not always the case. The smallest number with a spectrum witnessing this fact is $2673 = 3^5 \cdot 11$. The set of odd elements of the spectrum of 2673 is $\{1, 3, 9, 11, 27, 33\}$. It contains two primes, 3 and 11, both of them divide 33.

Example 5.8. The number $9261 = 3^3 \cdot 7^3$ is the smallest number such that its spectrum contains an odd prime and every odd prime in its spectrum is excessive. The set of odd elements of lspec(9261) is $\{1, 3, 7, 9, 21, 27, 49, 63\}$. It contains two primes, 3 and 7, both of them have excessive index 1. Let us explain how we found this example. Suppose $S = \mathrm{lspec}(n)$ has the required property. Then by Proposition 4.9, $E(S)$ must be a singleton. Also, $m := \max S_1$ cannot be a prime power, otherwise the prime of which m is a power will be non-excessive. So m has at least two prime factors, say $p_1 < p_2$. Since p_2 is excessive, therefore $p_2^2 \in S_1$. But that means $m > p_2^2 > p_1p_2$ and so $m = p_1p_2c$ for some odd number $c > 1$. By Lemma 4.1, the unique element $a \in E(S)$ is of the form p_1p_2d. Since it must be greater than $2m$, $d > 2c$. At this juncture, we make a guess: suppose $m = p_1^2p_2$ and $a = p_1p_2^2$. Then $n = (p_1p_2)^3$ and we need $p_2/p_1 = a/m > 2$. Minimizing n subjected to the inequality yields $p_1 = 3, p_2 = 7$ and so $n = (21)^3 = 9261$.

Now let n_0 be the smallest number such that its spectrum contains an odd prime and every odd prime in its spectrum is excessive. The argument above shows that $n_0 \leq 9261$ and is the product of two numbers of the forms p_1p_2c and p_1p_2d with $d > 2c$. Since $2(p_1p_2c)^2 = 2m^2 < n_0 \leq 9261$, $c^2 < 9261/2(15)^2$. Therefore, c must be 3 and $d \geq 7$. From this, we see that $p_1^2p_2^2 \leq 9261/21 = (21)^2$. If $p_1 = 3, p_2 = 5$, then $7 \leq d \leq 9261/(3 \cdot (15)^2)$, i.e. $d = 7, 9, 11$ or 13. But none of these choices produces a balanced spectrum. So $p_1 = 3, p_2 = 7$, this forces $d = 7$ and hence $n_0 = 9261$.

Example 5.9. The exponent β in Proposition 4.9 can be greater than 1. Let us find the smallest witness, say n_0, of this fact. First, let n be a number with $S = \mathrm{lspec}(n)$ witnessing $\beta > 1$. Let p be the unique excessive prime of

S and q be the largest non-excessive prime in S_1. For simplicity, let us write γ for γ_p. By Proposition 4.9 (i), $p > q$, $E(S) = \{p^{\gamma+1}, p^\gamma q^\beta\}$ and p does not divide $m := \max S_1$. Therefore, m is of the form $cq^{\beta+\kappa}$ where $\kappa \geq 0$ and c is not divisible by either p or q. The number of factors of the two elements in $mE(S)$ are both equal to the size of S; by equating them, we get $\beta\gamma = \kappa + 1$. Clearly, one solution of this equation with $\beta > 1$ is $\beta = 2, \gamma = \kappa = 1$. With these values, $E(S) = \{p^2, pq^2\}$. So the choices of p and q have to meet the following inequalities:

$$(q <)p < q^2 \text{ and } (m =)cq^3 < p^2/2. \tag{6}$$

Minimizing $n = cq^3p^2$ subjected to these inequalities yields $c = 1$, $p = 17$ and $q = 5$. So $n = 5^3 \cdot 17^2 = 36125$ and the exceptional set of its spectrum is $\{17^2, 17 \cdot 5^2\}$. It is indeed an example with $\beta(= 2) > 1$.

To show that n_0 is 36125, let us note that, from the argument above, n_0 is of the form $cq^{\beta+\kappa}p^{\gamma+1}$ with $cq^{\beta+\kappa} < p^{\gamma+1}/2$. Therefore, $n_0 > 2c^2q^{2(\beta+\kappa)}$. Since $\beta\gamma = \kappa + 1$ and $\beta \geq 2$, $36125 \geq n_0 > 2q^6$. That means $q \leq 5$. But one quickly rules out the possibility that $q = 3$, since no choice of p would satisfy the inequalities in (6). Hence $q = 5$, and it follows that the minimal choices for c and q are 1 and 17, respectively. This completes the proof.

To find an exceptional set with both γ and $\beta > 1$ will lead one to the number $21434375 = 5^5 \cdot 19^3$. The exceptional set of its spectrum is $\{19^3, 19^2 \cdot 5^2\} = \{6859, 9025\}$. An analysis similar to the one given above shows that 21434375 is indeed the smallest possible choice. We will leave the verification to the reader this time.

Finally, we give two algorithms answering the questions that we posed in the introduction. Algorithm 1 computes from a given number n its spectral class $L(n)$. Its correctness is guaranteed by Theorem 3.3, 3.4 and 3.7. Algorithm 2 computes, using Algorithm 1, from a finite set of numbers S the set of numbers with spectrum S. In particular, Algorithm 2 returns the empty set if S is not a spectrum. Here is the strategy: computes from S a number n such that $S = \text{lspec}(n)$ if S is a spectrum. The algorithm then returns either $L(n)$ or the empty set depending on whether the equality holds or not. We have implemented both algorithms using PARI/GP script.

Algorithm 1 Compute the spectral class of a natural number

Require: A natural number n
Ensure: $L(n)$ the spectral class of n
  $S := \text{lspec}(n)$; $S_i := \{a \in S \colon a \equiv i \pmod 2\}$ $(i = 0, 1)$.
  if $|S_0| = 0$ **then**
    $m_1 := \max S_1$; $\nu :=$ the least integer such that $2^{\nu+1} > m_1$.
    return $\{2^{\nu+i} m_1 \colon i \geq 0\}$.
  else if $|S_0| < |S_1|$ **then**
    return $\{n\}$.
  else if S is non-excessive **then**
    $m_0 := \max S_0$.
    return $\frac{1}{2} m_0 (P(S) \cup E(S))$.
  else
    $m_0 := \max S_0$.
    return $\frac{1}{2} m_0 E(S)$.
  end if

Algorithm 2 Compute the set of numbers with a given set as spectrum

Require: A finite set of natural numbers S.
Ensure: The set of natural numbers with length spectrum S.
  $S_i := \{a \in S \colon a \equiv i \pmod 2\}$ $(i = 0, 1)$.
  if $|S_0| = 0$ **then**
    $m_1 := \max S_1$; $\nu :=$ the least integer such that $2^{\nu+1} > m_1$; $n := 2^{\nu} m_1$.
  else if $|S_0| < |S_1|$ **then**
    $n := \frac{1}{2} \min S_0 \min D(S) \max D(S)$.
  else if $|E(S)| \neq 0$ **then**
    $m_0 := \max S_0$; $n := \frac{1}{2} m_0 \min E(S)$.
  else
    $m_0 := \max S_0$; $p :=$ the least prime $> m_0$; $n := \frac{1}{2} m_0 p$.
  end if
  if $\text{lspec}(n) = S$ **then**
    return $L(n)$.
  else
    return $\emptyset$.
  end if

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