LOCALLY FINITE HOMOGENEOUS GRAPHS

SHAWN HEDMAN AND WAI YAN PONG

ABSTRACT. A connected graph G is said to be z-homogeneous if any isomorphism between connected induced subgraphs of G extends to an automorphism of G. Finite z-homogeneous graphs were classified in [14]. We show that z-homogeneity is equivalent to finite-transitivity on the class of locally finite infinite graphs. Moreover, we classify the graphs satisfying these properties. Our study of bipartite z-homogeneous graphs leads to a new characterization for hypercubes.

October 3, 2009

1. Homogeneous Graphs

We study various notions of homogeneity for graphs. We begin by fixing some terminology that will be used throughout this article. The reader may consult [10] for undefined graph theoretic terms that appear here.

We view each graph G as a set of vertices with a binary edge relation that is both symmetric and irreflexive. In particular, all graphs in this paper are undirected graphs with neither loops nor multiple edges. If the edge relation holds for vertices a and b of graph G, then we refer to the set $\{a, b\}$ as an *edge* of G and say that a and b are *adjacent* in G or a and b are *neighbors* in G. We denote the set of vertices and the set of edges by $V(G)$ and $E(G)$, respectively. However, we often simply write G for $V(G)$. We let $|G|$ denote the cardinality of $V(G)$.

A *path* of length r from a to b in a graph G is a sequence of $r+1$ distinct vertices starting from a and ending with b such that consecutive vertices are adjacent. We speak of an (a, b)-path if we want to emphasis the start and the end point. We sometime speak of a path in G when we actually mean the subgraph of G consisting of the vertices of the path and the edges between consecutive vertices. A graph is *connected* if there is a path between any pair of vertices. For a connected graph G, the *distance* $\delta(a, b) := \delta_G(a, b)$ between two vertices a and b is the length of a shortest (a, b)-path. The interval $I(a, b)$ between a and b is the set of all vertices on shortest (a, b)-paths, that is,

$$I_G(a, b) := I(a, b) = \{c \in V : \delta(a, c) + \delta(c, b) = \delta(a, b)\}.$$

A *cycle* is a finite connected graph where every vertex has exactly two neighbors. More specifically, an n-cycle is a cycle with n vertices. By a *proper* cycle we mean a cycle with more than 3 vertices.

Let H and G be graphs. An *embedding* from H to G is an injective map f from $V(H)$ to $V(G)$ such that for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{f(a), f(b)\} \in E(G)$. A surjective embedding is an *isomorphism*. An *automorphism* of a graph G is an isomorphism of G to itself. A map from a subset of $V(H)$ to $V(G)$ is an *isometry* if $d_H(a, b) = d_G(f(a), f(b))$ for all $a, b \in H$. An isometry is necessarily an injective map.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is an *induced subgraph* if the inclusion of $V(H)$ into $V(G)$ is an embedding. An induced subgraph H of G is *convex* if $V(H)$ contains $I_G(a, b)$ for any $a, b \in H$. The intersection of convex subgraphs is convex and the *convex hull* of a set of vertices X in G is the smallest convex subgraph of G that contains X.

A graph G is a *tree* if and only if it is connected and has no cycles. A graph is *locally finite* if every vertex has finitely many neighbors. A graph is *e-regular* if each vertex has e neighbors. For any $e \geq 2$, the e-regular tree is a uniquely determined countably infinite graph. We shall call this graph the *e-tree* for short. The *m-clique*, denoted K_m, is the $(m-1)$-regular graph on m vertices.

Definition 1.1. *An* (m, n) *clique-tree* (for $n, m > 1$) is a graph G with the following properties:

- G is connected.
- G is $(m-1)n$-regular.
- Each vertex of G is the intersection of n distinct m-cliques. That is, given any vertex v there exist n disjoint sets of vertices $D_1, D_2, ..., D_n$ so that

for each i, $|D_i| = m - 1$ and the induced subgraph of G on $D_i \cup \{v\}$ is an m-clique. Moreover, there are no edges between D_i and D_j for $i \neq j$.

- G contains no proper cycle as induced subgraph.

We say that G is a *clique-tree* if it is an (m, n) clique-tree for some m and n.

It follows immediately from the definition above that every edge in a clique-tree belongs to a unique maximal clique. An easy consequence of this observation is that if a proper cycle is a subgraph of a clique-tree then it must be part of a clique. There is a more concise description of clique-trees in [11]. Our (m, n) clique-tree is the graph $X_{m-1,n}$ in [11].

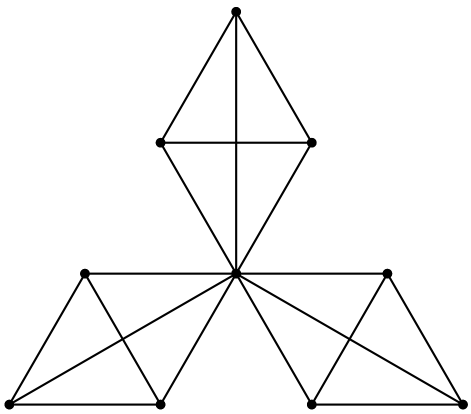

FIGURE 1. A portion of the (4,3) clique-tree

This paper was motivated by considering the homogeneity of e-trees. We show that various homogeneity properties possessed by e-trees are possessed by all clique-trees and by no other locally finite infinite graphs. Intuitively, the e-tree is homogeneous in the sense that every region of the graph looks the same. For example, Figure 2 depicts a subgraph of the 3-tree. This subgraph occurs everywhere in the 3-tree; the vertex labeled "a" could be any one of the graph's infinitely many vertices. To make the notion of homogeneity precise, we consider the action of the automorphism group on the graph G. Let $\mathrm{ISub}(G)$ denote the set of all induced

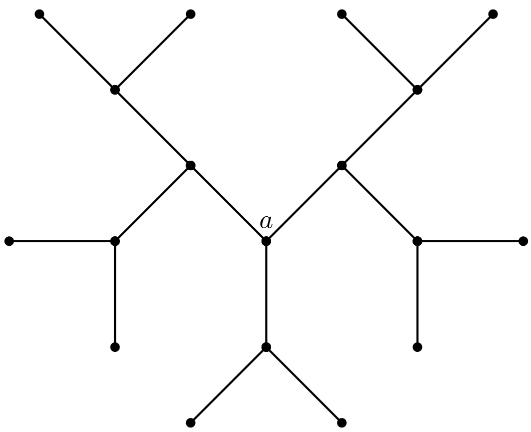

FIGURE 2. A portion of the 3-tree

subgraphs of G. For any $\mathcal{C} \subset \mathrm{ISub}(G)$, we say that G is $\mathcal{C}$-homogeneous if any isomorphism between members A and B of $\mathcal{C}$ extends to an automorphism of G. If G is $\mathcal{C}$-homogeneous where $\mathcal{C}$ is the class of all finite members of $\mathrm{ISub}(G)$, then G is said to be *ultra-homogeneous*. Trees do not possess this strong form of homogeneity. To see this, take A and B to be two pairs of non-adjacent vertices of different distances. We see that in a connected ultra-homogeneous graph, the maximum distance between vertices is 2. In particular, any infinite ultra-homogeneous connected graph has infinite degree. There are only countably many ultra-homogeneous graphs. Gardiner classified finite ultra-homogeneous graphs in [7]. Lachlan and Woodrow described the isomorphism types of countably infinite ultra-homogeneous graphs in [12].

To obtain a meaningful notion of homogeneity for locally finite infinite graphs such as the e-trees, we consider proper subsets $\mathcal{C}$ of $\mathrm{ISub}(G)$. To say that vertex a in Figure 2 could represent any vertex of the 3-tree is to say the 3-tree is *vertex-transitive*. This means the graph is $\mathcal{C}$-homogeneous where $\mathcal{C}$ is the set of all singletons. For e-trees, something much stronger is true.

Definition 1.2. *A graph G is said to be z-homogeneous if G is connected and is $\mathcal{C}$-homogeneous where $\mathcal{C}$ is the set of all finite connected graphs in* $\mathrm{ISub}(G)$.

The term "z-homogeneous" was introduced by Richard Weiss who classified all finite z-homogeneous graphs in [14]. Locally finite z-homogeneous graphs were determined by Gardiner [8] and Enomoto [6]. In particular, they showed that a locally finite infinite graph is z-homogeneous if and only if is it distance-transitive. Remarkably, Macpherson proved in [13] the locally finite distance-transitive infinite graphs are precisely the clique-trees. Gray and Macpherson have recently classified all (locally finite or not) countable z-homogeneous graphs (see [11]).

Definition 1.3. *A connected graph G is said to be k-transitive if, for any pair of isometric k-tuples of vertices of G, there is an automorphism mapping one to the other. In order words, any ordered k-tuples $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of vertices satisfying $\delta(x_i, x_j) = \delta(y_i, y_j)$ for all $1 \leq i, j \leq k$, there is an automorphism of G mapping x_i to y_i for each i.. We call G finitely-transitive if it is k-transitive for all $k \geq 1$.*

Distance-transitive graphs are hence the same as 2-transitive graphs. In [4], Cameron showed that every finite 6-transitive graph is finitely-transitive. This result followed from Cameron's classification of all finite 6-transitive graphs and the observation that each of these graphs is finitely-transitive. In light of these results, we have the following.

Theorem 1.4. *Let G be a locally finite infinite graph. The following are equivalent.*

(I) *G is 2-transitive.*
(II) *G is finitely-transitive.*
(III) *G is z-homogeneous.*
(IV) *G is a clique-tree.*

By the works of Macpherson [13], Gardiner [8] and Enomoto [6], we already know that (I), (III) and (IV) are equivalent for locally finite infinite graphs. Hence to prove Theorem 1.4 it suffices to verify that clique-trees are finitely-transitive. We will show this in section 2. In section 3, we restrict our attention to bipartite graphs and obtain a characterization for hypercubes without assuming the graph is finite.

Before giving the proofs, we note that the assumption that G is infinite is essential for Theorem 1.4. For example, the Petersen graph is z-homogeneous but not 3-transitive. The 4-cube (which we shall discuss in Section 3) is 3-transitive but not 4-transitive. We note too that Theorem 1.4 implies the equivalence of multiple notions of homogeneity on infinite locally finite graphs. For example, say that a graph G is *T-homogeneous* if it is $\mathcal{C}$-homogeneous where $\mathcal{C}$ is the set of trees in $\mathrm{ISub}(G)$. It is easy to see that T-homogeneity is a consequence of z-homogeneity and also that any T-homogeneous graph is 2-transitive. It follows that the notion of T-homogeneous is equivalent to these other notions on locally finite infinite graphs.

The authors would like to thank Dugald Macpherson for his helpful comments and for calling their attention to references [6] and [8].

2. Clique-Trees are finitely-transitive

We begin with the following lemma.

Lemma 2.1. *For any two vertices of a clique-tree there is a unique shortest path connecting them.*

Proof. Let G be a clique-tree. Suppose there are pairs of vertices of G invalidating the lemma. Choose a pair (u, v) with shortest distance among all the counter-examples. With this choice, any two paths connecting u and v share no vertices except u and v. So u and v are on a cycle and, since G is a clique-tree, it must be part of a maximal clique. Therefore, $\delta(u, v) = 1$. But this is a contradiction since our graphs have no multiple edges. $\square$

Note that in a clique-tree the set of vertices on the unique shortest path from u to v is simply $I(u, v)$ and the induced subgraph on $I(u, v)$ is $\mathrm{Conv}(\{u, v\})$. We write $\mathrm{Conv}(u, v)$ instead of $\mathrm{Conv}(\{u, v\})$ for simplicity. It is clear that a convex hull contains the interval of any of two of its vertices. And for a clique-tree, that is all a convex hull contains:

Proposition 2.2. *Let X be a set of vertices of a clique-tree. Then* $\mathrm{Conv}(X)$ *is the union of* $\mathrm{Conv}(a, b)$ *where a, b run through the vertices of X. In other words,* $\mathrm{Conv}(X)$ *is the induced subgraph on* $\bigcup_{a,b\in X} I(a, b)$.

Proof. We need to show that the union is convex. Suppose $u \in I(a_1, a_2)$ and $v \in I(a_3, a_4)$ where $a_1, a_2, a_3, a_4 \in X$ (not necessarily distinct). Clearly, we can assume at least three of the a_i's are distinct otherwise the result is trivial. We will show that $I(u, v) \subseteq I(a_i, a_j)$ for some $1 \le i, j \le 4$. Let γ be the unique shortest path from u to v. Let K be the unique clique (since we are in a clique-tree) containing the first edge of γ. Then either $I(a_1, u)$ or $I(u, a_2)$ intersects $V(K)$ at u only. Otherwise, there will be a shorter path from a_1 to a_2 bypassing u, contradicting $u \in I(a_1, a_2)$. So without loss of generality, we can assume $I(a_1, u) \cap V(K) = \{u\}$. Similarly, we can assume $I(a_3, v) \cap V(K') = \{v\}$ where K' is the unique clique containing the last edge of γ. We claim that $u, v \in I(a_1, a_3)$. This implies $I(u, v) \subseteq I(a_1, a_3)$ since $I(a_1, a_3)$ is convex. Suppose x is the vertex in $I(a_1, u) \cap I(a_1, a_3)$ that is closest to u and y is the vertex in $I(a_3, v) \cap I(a_1, a_3)$ that is closest to v. Unless $x = u$ and $y = v$, the union of $\mathrm{Conv}(x, u)$, $\mathrm{Conv}(u, v)$, $\mathrm{Conv}(v, y)$ and $\mathrm{Conv}(y, x)$ will be a cycle and hence must be part of a clique. In particular, x, y are vertices on $K = K'$ but this contradicts the assumption on either a_1 or a_3. Thus we establish the claim and hence the proposition. □

Lemma 2.3. *Clique-trees are* 3*-transitive.*

Proof. It is easy to see that z-homogeneous graphs are 2-transitive. In particular, clique-trees are 2-transitive.

Let G be a clique-tree and let $A = \{a_1, a_2, a_3\}$ be a set of three distinct vertices of G. Let d_i be the distance between a_j and a_k for $\{i, j, k\} = \{1, 2, 3\}$. We show that these three distances determine $\mathrm{Conv}(A)$ up to isomorphism.

For $\{i, j, k\} = \{1, 2, 3\}$, let y_i be the vertex in $I(a_i, a_j)$ and $I(a_i, a_k)$ furthest away from a_i. Consider $Y = \{y_1, y_2, y_3\}$. If $|Y| = 1$, then the three paths intersect at a single vertex. If $|Y| = 3$, then Y is a triangle (since no induced subgraph of G is a proper cycle). Clearly, $|Y|$ cannot equal 2.

For $i = 1, 2$, and 3, let $x_i = \delta(a_i, y_i)$, $\bar{x} = (x_1, x_2, x_3)$, $\bar{d} = (d_3, d_2, d_1)$, and $\bar{d_0} = (d_3 - 1, d_2 - 1, d_1 - 1)$. Let $M = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$.

If $|Y| = 1$, then $M\bar{x}^T = \bar{d}^T$. If $|Y| = 3$, then $M\bar{x}^T = \bar{d_0}^T$.

Given the distances d_1, d_2, and d_3, consider both $M^{-1}\bar{d}^T$ and $M^{-1}\bar{d_0}^T$. Since their difference is $(-1/2, -1/2, -1/2)^T$, exactly one of these two column vectors has integer entries and these must be equal to x_1, x_2 and x_3. Thus, the distances d_1, d_2, and d_3 determine whether $|Y| = 1$ or $|Y| = 3$ and also determine the distances x_i from a_i to y_i. This information determines the induced subgraph $\mathrm{Conv}(A)$ up to isomorphism. Thus we have shown that any isometry on A extends to an isomorphism on $\mathrm{Conv}(A)$. By z-homogeneity of clique-trees, this isomorphism can be extended to an automorphism of G. □

Theorem 2.4. *Clique-trees are finitely transitive.*

Proof. We have already shown that clique-trees are 3-transitive. Fix $k \geq 4$. Let G be a clique-tree and $f\colon a_i \mapsto b_i$ $(1 \leq i \leq k)$ be an isometry between two subsets of vertices of G of size k. We need to show that f extends to an automorphism of G.

For $i \leq k$, let $A_i = \{a_1, \ldots, a_i\}$ and $B_i = \{b_1, \ldots, b_i\}$. Let f_1 be the map sending a_1 to b_1. For $i \geq 1$, suppose by induction that we have built an isomorphism f_i from $\mathrm{Conv}(A_i)$ into $\mathrm{Conv}(B_i)$ such that $f_i(a_j) = b_j$ for all $1 \leq j \leq i$. We build f_{i+1} from f_i as follows. Pick any vertex c in $\mathrm{Conv}(A_{i+1})$ but not in $\mathrm{Conv}(A_i)$. Then by Proposition 2.2 $c \in I(a_{i+1}, a_j)$ for some $1 \leq j \leq i$. We set $f_{i+1}(c) = c'$ where c' be the unique vertex in $I(b_{i+1}, b_j)$ such that $\delta(a_{i+1}, c) = \delta(b_{i+1}, c')$. We need to argue that c' is independent of the choice of j. Suppose c is also on $I(a_{i+1}, a_m)$ for some m. Since G is 3-transitive (Lemma 2.3), there is an automorphism σ of G moving the triple (a_{i+1}, a_j, a_m) to (b_{i+1}, b_j, b_m). And σ induces an isomorphism between the convex hulls of these tuples. Hence c' must be in $I(b_{i+1}, b_m)$ as well. This shows that f_{i+1} is well defined.

By the definition of f_{i+1} and Proposition 2.2, f_{i+1} maps each $\mathrm{Conv}(a_{i+1}, a_j)$ isomorphically onto $\mathrm{Conv}(b_{i+1}, b_j)$ and hence f_{i+1} is an isomorphism (extending f_i) from $\mathrm{Conv}(A_{i+1})$ onto $\mathrm{Conv}(B_{i+1})$. By induction, we show that f can be extended to an isomorphism between the connected induced subgraphs $\mathrm{Conv}(A_k)$ and $\mathrm{Conv}(B_k)$ and hence by z-homogeneity of clique-trees, f extends to an automorphism of G. $\square$

3. A characterization of hypercubes

Let G be a locally finite graph and for $u \in G$, let $D_i(u)$ of the set of vertices of G of distance i from u. If the number of common neighbors of any pair of distance two vertices of G is a constant, then we denote this constant by c_2 otherwise we say that c_2 is undefined for G. At an early stage in our investigation of z-homogeneous graphs, we realize that:

Proposition 3.1. *If G is a locally finite z-homogeneous bipartite graph with regularity e, then a pair of distance 2 vertices of G have either $1, 2, e-1,$ or e common neighbors.*

Proof. Since a locally finite (connected) z-homogeneous graph is distance transitive, it is clear that $|D_1(u) \cap D_1(v)|$ is a constant for any pair (u, v) of distance 2 vertices. So c_2 is a number.

Take $v \in G$ and $w \in D_2(v)$. Let X be any subset of $D_1(v)$ of size c_2. Because G is bipartite there are no triangles and $X \cup \{v\} \cong (D_1(v) \cap D_1(w)) \cup \{v\}$. By z-homogeneity, there is an automorphism of G fixing v and taking $D_1(v) \cap D_1(w)$ to X. Let w_X be the image of w under this automorphism. If $X \neq D_1(v) \cap D_1(w)$, then $w \neq w_X$ since w is adjacent to no more than c_2 vertices in $D_1(v)$. For fixed $u \in D_1(v)$, there are exactly $\binom{e-1}{c_2-1}$ subsets of $D_1(v)$ of size c_2 that contain u. For each one of these subsets, there is a corresponding image of w. But there are only e vertices adjacent to c, one of which is a. So $\binom{e-1}{c_2-1} \leq (e-1)$. This inequality holds if and only if $c_2 - 1 = 0, 1, e-2,$ or $e-1$. $\square$

Certainly this is a rather weak result, especially now we know, thanks to the classification of Weiss and Theorem 1.4, precisely what are the locally finite z-homogeneous graphs. In particular, the bipartite ones are:

- the regular (infinite) trees.
- the even cycle C_{2n}, $n \geq 2$.
- the complete bipartite graphs, $K_{n,n}$, $n \geq 1$.
- the almost complete bipartite graphs $\tilde{K}_{n,n}$, i.e. $K_{n,n}$ with a perfect matching deleted.

We note that except for $c_2 = 2$ each of the possibilities for c_2 in Proposition p:c2 is realized infinitely often. For regular-trees and even cycles of length greater than 4, $c_2 = 1$. For complete bipartite graphs, c_2 is the regularity and for almost complete bipartite graphs, c_2 is one less than the regularity. But there are only two bipartite z-homogeneous graphs with $c_2 = 2$, namely the square $K_{2,2}$ and the cube $\tilde{K}_{4,4}$. Both graphs are hypercubes and hypercubes all have $c_2 = 2$. So we wonder if we can recover the family of hypercubes by replacing the z-homogeneous assumption with some weaker condition.

A *hypercube* is a graph with binary strings of a fixed length as vertices. Two vertices have an edge between them if and only if they differ in exactly one place. It is clear that hypercubes are regular. For $e \geq 1$, we denote the e-regular hypercube by Q_e. There are many ways to characterize hypercubes. The following is due to Mulder [2].

Theorem 3.2. *A connected bipartite e-regular graph is isomorphic to Q_e if and only if it has 2^e vertices and $c_2 = 2$.*

For the rest of this section, we will show that the global assumption on the size of the graph in 3.2 can be replaced by a mild convexity assumption, called *meshed*, on the distance function. A graph G is *meshed* if for any three vertices u, v, w with $\delta(v, w) = 2$, there exists a common neighbor x of v and w such that

$2\delta(u,x) \leq \delta(u,v) + \delta(u,w)$. We remark that weakly modular graphs are meshed. For more details on meshed graphs, the reader can consult [3].

Lemma 3.3. *Let G be a bipartite meshed graph. Suppose a, b, b' are vertices of G such that $\delta(a,b) = \delta(a,b')$ and $\delta(b,b') = 2$. Then there exists a common neighbor c of b and b' such that $\delta(a,c) = \delta(a,b) - 1$.*

Proof. Suppose b, b' are of distance m from a. By the definition of meshed graph, there exists a common neighbor c of b and b' such that $2\delta(a,c) \leq \delta(a,b) + \delta(a,b') = 2m$. So $\delta(a,c) \leq m$ but the inequality must be string since G is bipartite. Finally, the distance between a and c cannot be less than $m-1$ otherwise since c is neighbor of b (and b'), this contradicts the fact that b (and b') is of distance m from a. □

Theorem 3.4. *Let G be a connected bipartite e-regular meshed graph. Then $G \cong Q_e$ if and only if $c_2 = 2$.*

Proof. It is clear that $c_2 = 2$ for hypercubes. Conversely, suppose G is a connected e-regular meshed graph with $c_2 = 2$. By Theorem 3.2, it suffices to show that $|G| = 2^e$.

Fix $a \in G$. We show that $|D_i(a)| = \binom{e}{i}$ for each i. This is obvious for $i = 0$ or 1.

For any path u, v, w in G, there exists $v' \in G$ so that u, v', w is a path, $\{v, v'\}$ is not an edge of G, and v and v' are distinct. This is because $c_2 = 2$ and the graph is bipartite. We use this fact repeatedly and refer to the set $\{u, v, w, v'\}$ as a square.

Claim 1: Let $[D_1(u)]_2$ denote the set of two element subsets of $D_1(u)$ for arbitrary $u \in G$. There is a one-to-one correspondence between $D_2(u)$ and $[D_1(u)]_2$.

Proof of Claim 1: Given $c \in D_2(u)$, let $f(c) = D_1(u) \cap D_1(c)$. Since $c_2 = 2$, $f(c) \in [D_1(u)]_2$. To see that f maps onto $[D_1(u)]_2$, take $\{b, b'\}$ in $[D_1(u)]_2$. Because bub' is a proper path, there exists c such that $\{b, u, b', c\}$ is a square. Clearly, $c \in D_2(u)$ and $f(c) = \{b, b'\}$. Next, we show that the domain and range of f have the same size. Since each $b \in D_1(u)$ has degree e and no two vertices of $D_1(u)$ share an edge (since the graph is bipartite), each vertex in $D_1(u)$ is adjacent to $(e-1)$ vertices in $D_2(u)$. By assumption, each vertex of $D_2(u)$ is adjacent to $c_2 = 2$ vertices of $D_1(u)$. It follows that $|D_1(u)| \cdot (e-1) = |D_2(u)| \cdot 2$ which implies $|D_2(u)| = \binom{e}{2} = |[D_1(u)]_2|$. It follows that the function f from $D_2(u)$ onto $[D_1(u)]_2$ must be one-to-one.

We proceed by induction. Our induction hypothesis is that $|D_i(a)| = \binom{e}{i}$ for each i less than m for some $m \geq 3$. More specifically, for $1 \leq i < m$, we assume the following:

(1) each vertex of $D_i(a)$ is adjacent to exactly i elements of $D_{i-1}(a)$.
(2) each vertex of $D_i(a)$ is distance two from exactly $\binom{i}{2}$ vertices in $D_{i-2}(a)$.
(3) $|D_i(a)| = \binom{e}{i}$.

Fix $v \in D_m(a)$. Let $X = D_{m-1}(a) \cap D_1(v)$ and let $Y = D_{m-2}(a) \cap D_2(v)$. Let $x = |X|$ and $y = |Y|$. Our goal is to show that $x = m$, $y = m(m-1)/2$, and $|D_m(a)| = \binom{e}{m}$.

First, we show that $y = \binom{x}{2}$. Let $f : D_2(v) \to [D_1(v)]_2$ be as in the proof of Claim 1. Given $c \in Y$, there exists $b \in X$ so that c, b, v is a path. There must exist $b' \in X$ such that $\{c, b, v, b'\}$ is a square. On the other hand, given $\{b, b'\} \in [X]_2$, according to Lemma 3.3 there exists a common neighbor c of b and b' which is of distance $m-2$ from a. So $c \in D_2(v) \cap D_{m-2}(a) = Y$ and f is one-to-one. Therefore,

$$y = \frac{x(x-1)}{2}. \tag{1}$$

Consider next the correspondence between $D_m(a)$ and $D_{m-1}(a)$. Any vertex of $D_m(a)$ is adjacent to x vertices of $D_{m-1}(a)$. By assumption (i), every vertex of $D_{m-1}(a)$ is adjacent to $(m-1)$ vertices of $D_{m-2}(a)$. Since each vertex has degree e and since there are no edges within $D_{m-1}(a)$ (because G is bipartite), each vertex of $D_{m-1}(a)$ must be adjacent to $e-(m-1)$ vertices of $D_m(a)$. We conclude $|D_{m-1}(a)| \cdot (e-(m-1)) = |D_m(a)| \cdot x$. By assumption (iii), $|D_{m-1}(a)| = \binom{e}{m-1}$. Solving for $|D_m(a)|$ yields:

$$|D_m(a)| = \frac{e(e-1)...(e-(m-1))}{(m-1)!x} \tag{2}$$

Now consider the correspondence between $D_m(a)$ and $D_{m-2}(a)$. Each vertex of $D_m(a)$ is distance 2 from exactly y vertices of $D_{m-2}(a)$. If each vertex of $D_{m-2}(a)$ is distance 2 from z vertices of $D_m(a)$, then $|D_m(a)|y = |D_{m-2}(a)|z$ and so

$$|D_m(a)| = \frac{|D_{m-2}(a)|z}{y} = \frac{e(e-1)...(e-(m-3))z}{(m-2)!y} \tag{3}$$

We seek to determine the value of z.

Fix $c \in D_{m-2}(a)$. By Claim 1, there are $\frac{e(e-1)}{2}$ vertices of distance 2 from c. Some of these are in $D_{(m-4)}(a)$, some are in $D_{m-2}(a)$, and the remaining z of these vertices are in $D_m(a)$. By assumption (ii), the number of vertices in $D_{(m-4)}(a)$ of distance 2 from c is exactly $\frac{(m-2)(m-3)}{2}$.

To count the number of vertices in $D_{m-2}(a)$ of distance 2 from c, we count the number of paths c, d, c' with $d \in D_{m-1}(a)$ and $c' \in D_{m-2}(a)$. To justify this, we must show that for any $c' \in D_{m-2}(a)$ of distance 2 from c, there exists a unique $d \in D_{m-1}(a)$ adjacent to both c and c'. By Lemma 3.3, there exists $b \in D_{m-3}(a)$ adjacent to both c and c'. Because $c_2 = 2$, there is exactly one other vertex $d \in D_1(c) \cap D_1(c')$. Suppose for a contradiction that $d \in D_{m-3}(a)$. Then b and d are of distance 2 and are both in $D_{m-3}(a)$. By Lemma 3.3 again, there exists $c'' \in D_{m-4}(a)$ adjacent to both b and d. But then, $\{c, c', c''\} \subset D_1(b) \cap D_1(d)$ contradicting $c_2 = 2$.

By assumption (i), there are $(m-2)$ vertices of $D_{m-3}(a)$ adjacent to c. It follows that the other $e-(m-2)$ vertices adjacent to c must be in $D_{m-1}(a)$. So we have $e-(m-2)$ choices for d. Again by assumption (i), d is adjacent to $(m-1)$ vertices of $D_{m-2}(a)$. One of these is c, the other $(m-1)-1 = (m-2)$ of these are distance 2 from c. It follows that there are $[e-(m-2)](m-2)$ vertices in $D_{m-2}(a)$ distance two from a.

To summarize, there are $e(e-1)/2$ vertices of distance 2 from c. Of these, $(m-2)(m-3)/2$ are in $D_{m-4}(a)$, $[e-(m-2)](m-2)$ are in $D_{m-2}(a)$, and z are in $D_m(a)$. We have:

$$\frac{e(e-1)}{2} = (m-2)(m-3)/2 + [e-(m-2)](m-2) + z = \frac{(m-2)(2e-m+1)}{2} + z.$$

And so:

$$z = \frac{e(e-1)}{2} - \frac{(m-2)(2e-m+1)}{2} = \frac{(e-(m-2))(e-(m-1))}{2}.$$

Substitution this into Equation 3 yields:

$$|D_m(a)| = \frac{|D_{m-2}(a)|z}{y} = \frac{e(e-1)...(e-(m-3))(e-(m-2))(e-(m-1))}{(m-2)!2y}$$

from which we obtain:

$$(m-1)x = 2y.$$

Comparing this with Equation (1), we see that $x = m$ and $y = m(m-1)/2$ as we wanted to show. It follows from either Equation (2) or (3) that $|D_m(a)| = \binom{e}{m}$. By induction, $|D_i(a)| = \binom{e}{i}$ for all i as claimed. Finally, $|G| = \Sigma_i |D_i(a)| = 2^e$ and $G \cong Q_e$ by Theorem 3.2. $\square$

We end this article by remarking that the assumption that G is a meshed graph in Theorem 3.4 cannot be removed. For example, the incident graph of the unique Hadamard 2-(11,5,2) design is a bipartite 5-regular (in fact distance transitive) graph with $c_2 = 2$. We will give a realization of the graph here but refer the reader to [10, Chapter 5], [1] and [9] for more details. This graph has 22 vertices. The vertices in one partition are called points, say $0, 1, \ldots, 9, A$ and the vertices of the other partition are called lines, they are 5-element sets of points. Here are the 11 lines:

01368, 01479, 02569, 0248A, 0357A, 12345, 1267A, 1589A, 23789, 3469A, 45678.

There is an edge between a point and a line if the point is a member of the line.

References

[1] M.R. Alfuraidan; J.I. Hall, Smith's Theorem and a characterization of the 6-cube as distance-transitive graph, Journal of Algebraic Combinatorics, Volume 24, Number 2, (2006) p.195-207.

[2] A.S. Asratian, T.M.J. Denley, and R. Häggkvist, *Bipartite Graphs and their Applications*, Cambridge University Press, 1998.

[3] H. J. Bandelt, V. Chepoi, Metric graph theory and geometry: a survey, Contemporary Mathematics, Vol. 453 (2008), pp. 49-86.

[4] P.J. Cameron, 6-transitive graphs, *J. Comb. Theory*, Ser. B, **28** (1980), 168-179.

[5] P.J. Cameron, Automorphism groups of graphs, in *Selected Topics in Graph Theory 2*, Academic Press, 1983.

[6] H. Enomoto, Combinatorially homogeneous graphs. *J. Comb. Theory* Ser. B, **30**(2), 1981, 215-223.

[7] A. Gardiner, Homogeneous graphs, *J. Comb. Theory*, Ser. B, **20** (1976), 94-102.

[8] A. Gardiner, Homogeneity conditions in graphs. *J.Comb. Theory*, Ser. B, **24**(3) (1978), 301-310.

[9] A. Gardiner; C.E. Praeger, Distance-Transitive graphs of valency five, Proceedings of the Edinburgh Mathematical Society, **30** (1987), 73-81.

[10] C. Godsil; G. Royle, Algebraic Graph Theory. Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.

[11] R. Gray; H.D. Macpherson, Countable connected-homogeneous graphs, preprint.

[12] A.H. Lachlan; R.E. Woodrow, Countable homogeneous undirected graphs, *Trans. Amer. Math. Soc.*, **262** (1980), 51-94.

[13] H.D. Macpherson, Infinite distance-transitive graphs of finite valency, *Combinatorica*, **2**, (1982), 63-70.

[14] R. Weiss, Glatt einbettbare untergraphen, *J. Comb. Theory*, Ser. B, **21** (1976), 276-281.

[15] H.P. Yap, *Some Topics in Graph Theory*, Cambridge University Press, 1986.

Department of Mathematics and Computer Science, Florida Southern College

E-mail address: shedman@flsouthern.edu

Department of Mathematics, California State University Dominguez Hills

E-mail address: wpong@csudh.edu