On Lascar rank and Morley rank of definable groups in differentially closed fields

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Abstract

Morley rank and Lascar rank are equal on generic types of definable groups in differentially closed fields with finitely many commuting derivations.

It is proved in [2] that the Morley rank (RM) and the Lascar U-rank (RU) of a complete type in the theory of differentially closed fields of characteristic 0 (DCF_0) can be different. In contrast, a result in [7, Theorem 2.7] states that in DCF_0 when the Morley rank of a complete type is a limit ordinal then so is its U-rank and these two are equal. However, this result is not very useful in computing ranks since most of the time it is the Morley rank that is hard to compute. With this in mind, it is thus desirable to find other (maybe more useful) conditions on types to guarantee that these two ranks agree. We will show that one such condition is for the type to be a generic type of a definable group in DCF_0. We will present our proof in the following way: First we show that Morley rank and U-rank are equal for definable groups in an ω-stable theory satisfying certain conditions. Then we verify these conditions for the theory of differentially closed fields with finitely many, say m, commuting derivations (m-DCF_0). An example is given in Appendix A showing that in general RM and RU are not equal for ω-stable groups. Here by a group we mean a group with possible additional structures.

Preliminaries

Let T be an ω-stable theory. By the Morley rank of T we mean the Morley rank of the formula $x = x$ calculated in some saturated model of T. Let φ be a formula in T^{eq} with parameters. The U-rank of φ is the supremum of $\mathrm{RU}(p)$ where p runs through the complete types in T^{eq} containing φ. If X is the set defined (in some big model) by φ, we write either $\mathrm{RU}(X)$ or $\mathrm{RU}(\varphi)$ for this supremum. Let G be an ω-stable group. A formula φ in G is generic if finitely many translates of φ cover G. A type p in G is generic if every formula

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in p is generic. Also let us recall that the set of types in G of maximal U-rank, of maximal Morley rank and the set of generic types of G are all the same. Moreover if G is connected, it has a unique generic type over any set of parameters over which G is defined. Next, let us summarize the major results that will be used in our proofs.

Fact 1 *Let G be an ω-stable group. If $\mathrm{RU}(G)$ is finite, then the U-rank of any complete type in G^{eq} equals its Morley rank. In particular, $\mathrm{RM}(G) = \mathrm{RU}(G)$.*

A proof of this fact can be found in [4, p. 462]. We take the opportunity to give a proof of a more general statement in Appendix B yielding also uniform bounds for Morley degree. Here we want to point out that Fact 1 fails for non-generic types of G, even in DCF_0, if G has infinite Morley rank. The example given by Hrushovski and Scanlon in [2] shows that there are types with U-rank and Morley rank different even though the ranks of the types themselves are finite. So to see that Fact 1 does not hold in full generality, one can simply take G to be the additive group of affine n-space (for suitable n).

The next fact is a result of McGrail [5, Theorem 5.2.2] which generalizes Theorem 2.7 of [7] to the several derivations case. Here we only state it in the form that we need.

Fact 2 *Let p be a complete type in m-DCF_0 of infinite U-rank. Then there exist positive integers d and e such that*

$$\omega^e d \leq \mathrm{RU}(p) \leq \mathrm{RM}(p) \leq \omega^e(d+1).$$

The next two facts are due to Berline and Lascar [1, §III Corollary 8.2 and §IV.Corollary 2.7].

Fact 3 (*Lascar inequalities for definable groups*)
Let G be a definable group and H be a definable subgroup of G then:

$$\mathrm{RU}(H) + \mathrm{RU}(G/H) \leq \mathrm{RU}(G) \leq \mathrm{RU}(H) \oplus \mathrm{RU}(G/H).$$

Fact 4 *Let G be a superstable group. If $\mathrm{RU}(G) = \omega^{\alpha_1}k_1 + \cdots \omega^{\alpha_n}k_n$ (in Cantor normal form, i.e. $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ and the k_i's are integers ≥ 1), then G has definable normal subgroups of any U-rank of the form $\omega^{\alpha_1}k_1 + \cdots \omega^{\alpha_r}k_r, r \leq n$.*

1 The Result

Theorem 1.1. *Let T be an ω-stable theory of Morley rank ω^α. Suppose for any ordinal $1 \leq \tau \leq \alpha$ and integer $d \geq 1$, T^{eq} satisfies the condition:*

$$\textit{For any complete type } p, \textit{if } \mathrm{RM}(p) = \omega^\tau d \textit{ then } \mathrm{RU}(p) = \mathrm{RM}(p). \qquad (*)$$

Then Morley rank and U-rank agree for definable groups in T^{eq}.

We divide the proof of Theorem 1.1 into several steps. Let G be a definable group in T^{eq}. We can assume G has nonzero U-rank otherwise the conclusion of the theorem is trivial. Since G and its connected component have the same Morley rank and U-rank, we may and will assume that G is connected. Now, let us start by proving a special case of Theorem 1.1.

Proposition 1.2. *Under the assumptions of Theorem 1.1, if* $\mathrm{RU}(G) = \omega^\tau d$ *then* $\mathrm{RM}(G) = \mathrm{RU}(G)$.

Proof. The proposition is simply Fact 1 when $RU(G)$ is finite. So suppose both d and $\tau > 0$. Since $\mathrm{RM}(G) \geq \mathrm{RU}(G) = \omega^\tau d$, there is a definable subset X of G of Morley rank $\omega^\tau d$. Let p be a type in X of the same Morley rank. By condition (*), p has U-rank $\omega^\tau d$ as well and hence is the generic type of G. But this means that any definable subset of G of Morley rank $\omega^\tau d$ contains the generic type and hence no two of them are disjoint. Therefore $\mathrm{RM}(G) = \omega^\tau d$. □

Since the U-rank of T is at most ω^α, it follows easily from the Lascar inequalities that $\mathrm{RU}(G)$ is of the form $\omega^{\tau_k} d_k + \omega^{\tau_{k-1}} d_{k-1} + \cdots + \omega^{\tau_1} d_1$ where $\alpha \geq \tau_k > \tau_{k-1} > \cdots > \tau_1$ are ordinals and d_i's are positive integers. By Fact 4, G has a definable (normal) subgroup, say H, of U-rank $\omega^{\tau_k} d_k$. Since H is normal in G, so is H° its connected component. However, the normality of H in G will not play a role in the following arguments. In any case, replacing H by H°, we assume that H is connected as well. Note that $\mathrm{RU}(G/H) < \omega^{\tau_k}$ otherwise by Fact 3, $\mathrm{RU}(G) \geq \omega^{\tau_k}(d_k + 1)$ contradicting our assumption. Now since $\mathrm{RU}(G/H) < \omega^{\tau_k}$, $\mathrm{RU}(H) + \mathrm{RU}(G/H) = \mathrm{RU}(H) \oplus \mathrm{RU}(G/H)$. Therefore, the Lascar inequalities imply $\mathrm{RU}(G) = \mathrm{RU}(H) + \mathrm{RU}(G/H)$ and hence

$$\mathrm{RU}(G/H) = \omega^{\tau_{k-1}} d_{k-1} + \cdots + \omega^{\tau_1} d_1.$$

Let Y be a definable subset of G. Denote by Y^* the set

$$\{a \in G : \mathrm{RM}(Y \cap aH) = \mathrm{RM}(H)\}.$$

Note that Y^* is the set of all $a \in G$ such that the formula defining $a^{-1}Y$ is in the generic type of H. Thus by definability of types, Y^* is definable.

We would like to thank Françoise Delon and the referee for showing us how to simplify our proof of the following Lemma. This enables us to generalize the original version of Theorem 1.1 to the form as it is stated now.

Lemma 1.3. *Let* Y *be a definable subset of* G *with* $\mathrm{RM}(Y) \geq \mathrm{RM}(H) + \gamma$ *then* Y^*/H *has Morley rank at least* γ.

Proof. We prove this by induction on γ. Suppose $\gamma = 0$. All we have to show is that Y^* (hence Y^*/H) is nonempty. Let X be a definable subset of Y of Morley rank $\omega^{\tau_k} d_k$. By adding parameters, we can assume the sets that we have considered are definable over the empty set. Let a be a generic point of X (over $\emptyset$), i.e. $\mathrm{RM}(a) = \mathrm{RM}(X) = \omega^{\tau_k} d_k$. By condition (*), $\mathrm{RU}(a) = \omega^{\tau_k} d_k$ as well. Since aH (as a point in G/H) is definable from a, the Lascar inequalities give:

$$\mathrm{RU}(a/aH) + \mathrm{RU}(aH) \leq \mathrm{RU}(a) = \mathrm{RU}(a, aH) \leq \mathrm{RU}(a/aH) \oplus \mathrm{RU}(aH).$$

But $\mathrm{RU}(aH) \leq \mathrm{RU}(G/H) < \omega^{\tau_k}$, so $\mathrm{RU}(a/aH) = \omega^{\tau_k} d_k$ and $\mathrm{RU}(aH) = 0$ by the above inequalities. It follows that $Y \cap aH$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and since $a \in Y \cap aH$, we have $\mathrm{RM}(Y \cap aH) = \omega^{\tau_k} d_k$. Therefore $a \in Y^*$.

When γ is a limit ordinal, the statement follows from the induction hypothesis. Now suppose the assertion is true for γ and $\mathrm{RM}(Y) \geq \mathrm{RM}(H) + \gamma + 1$. Then there are $Z_1, Z_2, \ldots$ infinitely many pairwise disjoint definable subsets of Y of Morley rank $\mathrm{RM}(H) + \gamma$. By the induction hypothesis we have $\mathrm{RM}(Z_i^*/H) \geq \gamma$ for each i. Note that these Z_i^*/H's are pairwise disjoint subsets of Y^*/H: otherwise two of the Z_i's, which are themselves disjoint, will intersect the same coset of H generically. This contradicts the fact that H is connected. So we have $\mathrm{RM}(Y^*/H) \geq \gamma + 1$. □

Proof of Theorem 1.1. Recall that $\mathrm{RU}(G) = \omega^{\tau_k} d_k + \omega^{\tau_{k-1}} d_{k-1} + \cdots + \omega^{\tau_1} d_1$. We prove the statement by induction on the number of terms in this expression. The case $k = 1$ follows from Proposition 1.2. Suppose $k > 1$, by the induction hypothesis and Proposition 1.2, U-rank and Morley rank agree for both G/H and H. If $\mathrm{RM}(G) > \mathrm{RU}(G)$ then it has the form $\mathrm{RM}(H) + \gamma$ where $\gamma > \mathrm{RM}(G/H)$. Apply Lemma 1.3 to the case $Y = G$, we have $\mathrm{RM}(G/H) \geq \mathrm{RM}(G^*/H) \geq \gamma$ which is a contradiction. This finishes the proof. □

Corollary 1.4. *Morley rank equals U-rank for definable groups in m-DCF_0.*

Proof. By [5, Theorem 5.2.10], the Morley rank of m-DCF_0 is ω^m. Condition (*) in Theorem 1.1 follows immediately from Fact 2. So we are done. □

A An ω-stable group with $\mathrm{RM} \neq \mathrm{RU}$

The following is an easy example of an ω-stable group $(G, +, \cdots)$ which has U-rank ω but Morley rank strictly greater than ω. We give it here anyway for the record. We build G in three steps, all well-known constructions:

Step 1. For $0 \leq n < \omega$, build a countable structure M_n of Morley rank $n+1$ but U-rank 1 (that is all complete 1-types in a big model have U-rank at most 1). M_n will have unary predicates P_η for $\eta \in \omega^{\leq n}$ (where $\omega^{\leq n}$ denotes sequences from ω of length at most n). $P_\emptyset$ will be all of M_n. If η is an initial segment of τ then P_η contains P_τ.If η is not an initial segment of τ then P_η and P_τ are disjoint. All P_η are infinite. M_n clearly satisfies the requirements.

Step 2. Build a countable structure M of Morley rank ω but with all complete 1-types of U-rank ≤ 1. Simply let M be the disjoint union of the M_n's, where we chose disjoint languages for the M_n's, and where we adjoin a new predicate Q_n say for each M_n.

Step 3. Build a group G (that is G is equipped with a group operation $+$ as well as additional structure). Let V be an infinite-dimensional vector space over $\mathbb{Q}$ (in the module language or even just the language of groups). Let I be a $\mathbb{Q}$-basis for V. Adjoin a predicate R for I, and adjoin unary predicates such that the structure consisting of R with these unary predicates is isomorphic to M from Step 2. Let G be the vector space V equipped with $+$, R and all these unary predicates.

Let G^* be a saturated model, and R^* the interpretation of R. Then R^* has U-rank 1 (all non-algebraic 1-types in R^* have U-rank 1) and Morley rank ω in G^*. Likewise $R^* + \cdots + R^*$ (n-times) has U-rank n and Morley rank ω^n in G^*. The generic type of G^* (saying x is not a sum of elements of R^*) has U-rank ω. G^* has Morley rank ω^ω.

B Morley degree in finite-dimensional and finite Morley rank theories

We give a proof of the following result:

Proposition B.1. *Let T be a complete stable theory (in a first order language L). Assume that there is a fixed set of strongly minimal sets, D_i for $i \in I$, such that any (non-algebraic) type is nonorthogonal to some D_i. Then*

(i) *Morley rank = U-rank $< \omega$ (i.e. each complete type has finite U-rank which equals its Morley rank).*

(ii) *For any L-formula $\phi(x,y)$, there are $m_1, .., m_r < \omega$ and L-formulas[1] $\psi_1(y), .., \psi_r(y)$ partitioning y-space such that for each i, and b,*

$$\mathrm{RM}(\phi(x,b)) = m_i \textit{ iff } \models \psi_i(b),$$

moreover there is a uniform finite bound on the Morley degrees of the $\phi(x,b)$'s.

Remark B.2. Before giving the proof, let us remark the following:

(i) The assumptions of Proposition B.1 easily imply that T is superstable and every complete type has finite U-rank. (For example, any type will be analyzable in finitely many steps in the set of D_i.)

(ii) The proof below works equally well for many-sorted T that satisfies the assumptions of Proposition B.1. In the 1-sorted case, it is easy to see that T is finite-dimensional: there is a finite subset $\{D_1, \ldots, D_s\}$ of the set of D_i's such that any non-algebraic type is non-orthogonal to at least one D_i ($1 \leq i \leq s$). A quick proof suggested by the referee is as follows: Fix a set of parameters A over which all the D_i's are defined. By stability, whenever $A \subseteq \mathcal{M} \precneqq \mathcal{N}$ are models of T, there is some D_i realized in $\mathcal{N} \setminus \mathcal{M}$. This implication can be formalized by adding a predicate P for an elementary submodel. Applying compactness gives the finite set $D_1, \ldots, D_s$.

[1] Here L-formulas are formulas in the language L without parameters.

(iii) By a result of Lascar [4], any theory of a group G of finite Morley rank satisfies the hypothesis of the proposition. Thus, there is a bound on the Morley degrees (and thus on the index of the connected components) of any uniformly definable family of subgroups of G. Moreover, any type in G^{eq} (not only the generic type) has U-rank equal to Morley rank.

The results above should be considered well-known and belonging essentially to model-theoretic antiquity, the early 1970's, when Baldwin and Zilber independently proved the finiteness of Morley rank in uncountably categorical theories. However, possibly because of the subtlety of the issues involved, the full content and implications of the results have remained a little obscure. For example the status of Remark B.2 (iii) was unclear to quite a few people. Also in the first author's book, Geometric Stability Theory [6], a mistake was made precisely on these issues: the proof of Lemma 1.5.12 (equality of Morley and Lascar rank in totally transcendental unidimensional theories) had an error, pointed out by Enrique Casanovas. So we take the liberty to give a somewhat detailed proof of Proposition B.1.

We assume T is as in the hypothesis of the theorem. Work in a big model of T. The symbols $x, y, z, \ldots$ range over arbitrary finite sequences of variables when they appear in formulas. However, for simplicity we will use them as place holders for parameters for types as well. Also, when we write a complete type over b as $p(x, b)$ we are implicitly assuming that $p(x, y)$ is a complete type over the empty set, i.e. $p(x, y)$ is precisely the type $tp(a, b/\emptyset)$ for any a realizing $p(x, b)$.

Lemma B.3. *For any $n < \omega$.*

$(a)_n$ *Let $p(x, b)$ a complete type of U-rank n. Then*

(i) *$p(x, b)$ has Morley rank n, and*

(ii) *there are formulas $\phi(x, y) \in p(x, y)$, $\psi(y) \in tp(b)$, and there is $k < \omega$ such that for any b',*

$$\text{if } \models \psi(b') \text{ then } \phi(x, b') \text{ has Morley rank } n \text{ and Morley degree at most } k. \qquad (\dagger)$$

Moreover for any formula $\chi(x, y) \in p(x, y)$, we can choose $\psi(y)$ and $\phi(x, y)$ so that $\phi(x, y)$ implies $\chi(x, y)$ and ($\dagger$) holds.

$(b)_n$ *Let $\rho(x)$ be a fixed formula over a parameter set A, with* $\mathrm{RM}(\rho(x)) = n + 1$. *Let $\phi(x, y)$ be any L-formula (or even L_A-formula). Then for each $i \leq n + 1$ there is some formula $\psi_i(y)$ over A, and some $k_i < \omega$ such that*

(i) *the formulas $\psi_i(y)$ partition y-space, and*

(ii) *for each i, $\psi_i(b')$ holds if and only if $\phi(x, b') \wedge \rho(x)$ has Morley rank i and Morley degree at most k_i.*

Proof. We assume for simplicity that the D_i's are defined over the empty set, as routine arguments allow us to add parameters. The statements $(a)_n$ and $(b)_n$ are proved by induction on n. In fact our proofs show these will hold after adding any parameters we want. Let us first look at the case when $n = 0$. The statements in $(a)_0$ are immediate except maybe the last sentence: Let $\phi'(x, y)$ be such that $\phi'(x, b)$ is an algebraic formula in $p(x, b)$. For any $\chi(x, y) \in p(x, y)$, take $\phi(x, y)$ to be $\phi'(x, y) \wedge \chi(x, y)$. Then $\phi(x, b)$ is in $p(x, b)$ and still algebraic. Hence for some k the formula $\exists^{\leq k} x\ \phi(x, y)$ is in $tp(b)$. This finishes $(a)_0$. For $(b)_0$, we may assume that $\rho(x)$ is strongly minimal. We have to show that for any formula $\phi(x, y)$, there is some $k < \omega$ such that for any b', $\phi(x, b') \wedge \rho(x)$ has finitely many realizations if and only if it has at most k realizations. This is immediate by strong minimality of $\rho(x)$. Let us now assume $(a)_i$ and $(b)_i$ for all $i \leq n - 1$.

Proof of $(a)_n$. Suppose $\mathrm{RU}(p(x, b)) = n > 0$. So by the assumptions there is some finite tuple c, some realization a of a nonforking extension of $p(x, b)$ over $\{b, c\}$, and d in some D_i such that $d \in acl(a, b, c) \setminus acl(b, c)$. Let $\rho(x, y, z, w)$ be an L-formula such that $\rho(a, b, c, w) \in tp(d/abc)$ and for any a', b', c', $\rho(a', b', c', w)$ has at most k' solutions in D_i (for suitable $k' < \omega$). By the finite U-rank equality, $\mathrm{RU}(a/bcd) = n - 1$. By $(a)_{n-1}$, $\mathrm{RM}(a/bcd) = n - 1$ and we find formulas $\phi(x, y, z, w)$ in $r(x, y, z, w) = tp(a, b, c, d)$, $\psi(y, z, w)$ in $tp(b, c, d)$ and $k < \omega$ such that $\phi(x, b', c', d') \wedge \rho(x, b', c', d')$ has Morley rank $n - 1$ and Morley degree at most k, for all (b', c', d') satisfying ψ. As $d \notin acl(b, c)$ and D_i is strongly minimal, there is $\tau(y, z) \in tp(b, c)$ such that $\tau(b', c')$ implies that for infinitely many $d' \in D_i$,

$$\exists x\ (\phi(x, b', c', d') \wedge \rho(x, b', c', d') \wedge \psi(b', c', d')).$$

Let $\theta(x, y, z)$ be the formula

$$\exists w \in D_i(\phi(x, y, z, w) \wedge \rho(x, y, z, w) \wedge \tau(y, z) \wedge \psi(y, z, w)).$$

Claim. For any b', c', if $\theta(x, b', c')$ is consistent then it has Morley rank n and Morley degree at most k.

Proof of Claim. We will fix b', c' for which $\theta(x, b', c')$ is consistent. Let X be the set defined by $\theta(x, b', c')$. For $d' \in D_i$, let $Y_{d'}$ be the set defined by $\phi(x, b', c', d') \wedge \rho(x, b', c', d')$. So (after throwing finitely many points away from D_i) our assumptions imply that $Y_{d'}$ has Morley rank $n - 1$ and Morley degree at most k for all $d' \in D_i$ and that X is the union of the $Y_{d'}$'s as d' ranges over D_i. Note that, by the properties of ρ, any intersection of more than k' of the $Y_{d'}$'s is empty, easily yielding that X has Morley rank at least n.

Subclaim. Let Z be a definable subset of X such that $\mathrm{RM}(Z) = n$. Then for all but finitely many $d' \in D_i$, $\mathrm{RM}(Z \cap Y_{d'}) = n - 1$.

Proof of Subclaim. First note that $\mathrm{RM}(Z \cap Y_{d'}) \leq n - 1$ for all $d' \in D_i$. By $(b)_{n-1}$, the set

$$W = \{d' \in D_i : \mathrm{RM}(Z \cap Y_{d'}) = n - 1\}$$

is definable, so is finite or cofinite. Suppose for a contradiction that it is finite. So Z_1, the union of the $Z \cap Y_{d'}$'s for $d' \in W$, has Morley rank $n-1$. Let Z_2 be the union of the $Z \cap Y_{d'}$'s for $d' \in D_i \setminus W$. Then $Z = Z_1 \cup Z_2$. We will show that Z_2 has Morley rank at most $n-1$, yielding a contradiction. Note that Z_2, being a subset of Z, has Morley rank at most n, so let $a' \in Z_2$ be chosen such that $tp(a')$ (over all parameters involved so far, amongst which b' and c') has Morley rank equal to the Morley rank of $Z_2 = m$ say. Let $\mathrm{RU}(a') = m' \leq m$ and $d' \in D_i \setminus W$ such that $a' \in Y_{d'}$. As $d' \in acl(a')$, $\mathrm{RU}(a', d') = \mathrm{RU}(a')$. As $\mathrm{RM}(Z \cap Y_{d'}) < n-1$, $m'' = \mathrm{RU}(a'/d') < n-1$. The finite U-rank equality yields that $\mathrm{RU}(a') = m' \leq m'' + 1 \leq n-1$. The induction hypothesis $(a)_{m'}$ implies that $m = m' \leq n-1$. Thus we have shown that $\mathrm{RM}(Z_2) \leq n-1$. So $\mathrm{RM}(Z) \leq n-1$, a contradiction. The subclaim is established.

It follows from the subclaim and the assumptions on the $Y_{d'}$ $(d' \in D_i)$ that there are at most k pairwise disjoint definable subsets of X of Morley rank n. Thus X has Morley rank n and Morley degree at most k, proving the claim.

Now any $\chi(x,y,z) \in tp(a,b,c)$ can be viewed as a formula in $tp(a,b,c,d)$. By $(a)_{n-1}$, we can choose $\psi'(y,z,w) \in tp(b,c,d)$ such that (†) holds for ψ' and $\phi \wedge \rho \wedge \chi$. Let $\tau'(y,z)$ and $\theta'(y,z,w)$ be the formulas obtained by replacing ψ by ψ' in the constructions of τ and θ. Finally let $\mu(y,z)$ be the formula $\exists w \in D_i\ \psi'(y,z,w) \wedge \tau'(y,z)$. Then from the constructions, it follows that $\theta'(x,y,z)$ implies $\chi(x,y,z)$ and by the Claim, (†) holds for the pair $\mu(y,z)$ and $\theta'(x,y,z)$. Therefore $(a)_n$ is true for the type $tp(a/bc)$.

As $tp(a/bc)$ does not fork over b, $p(x,b)$ has Morley rank n, yielding $(a)_n(i)$ for $p(x,b)$. We have to get $(a)_n(ii)$ for $p(x,b)$. Let $\beta(x,b)$ be any formula in $p(x,b)$. Then $\beta(x,b) \in tp(a/bc)$ and by what we have proved above there is $\theta(x,b,c) \in tp(a/bc)$ and $\mu(y,z) \in tp(b,c)$ such that $\theta(x,b,c)$ implies $\beta(x,b)$ and for all (b',c') satisfying $\mu(y,z)$, $\theta(x,b',c')$ has Morley rank n and Morley degree at most k for some fixed k. Let $\mu'(y,z)$ be $\mu(y,z) \wedge \forall x(\theta(x,y,z) \rightarrow \beta(x,y))$. By the Open Mapping Theorem, there is a formula $\eta(x,b)$ such that for any complete type $q(x)$ over b, $\eta(x,b) \in q(x)$ if and only if some nonforking extension of q over bc contains $\theta(x,b,c)$. Moreover, by Lemma 2.2 in [6], $\eta(x,b)$ is equivalent to a positive Boolean combination of b-conjugates of $\theta(x,b,c)$. Hence $\eta(x,b)$ is equivalent to a *disjunction* $\bigvee_{i=1}^{r}(\theta(x,b,c_i) \wedge \eta(x,b))$ where the c_i realize $tp(c/b)$. Note also that $\eta(x,b)$ implies $\beta(x,b)$. Again by what we have proved above for $tp(a/bc)$ there are formulas $\theta'(x,b,c) \in tp(a/bc)$ and $\mu''(y,z) \in tp(b,c)$ such that $\mu''(b',c')$ implies that $\theta'(x,b',c')$ has Morley rank n. Also we may assume that $\mu''(y,z)$ implies $\mu'(y,z)$ and $\forall x(\theta'(x,y,z) \rightarrow (\theta(x,y,z) \wedge \eta(x,y)))$. It then follows that

if (b',c') satisfies $\mu''(y,z)$ then $\theta(x,b',c') \wedge \eta(x,b')$ has Morley rank n and Morley degree at most k. (**)

Let $\epsilon(y)$ be the formula

$$\exists z_1, \ldots, z_r \left(\bigwedge_{i=1}^{r} \mu''(y,z_i) \wedge \forall x \Big(\eta(x,y) \leftrightarrow \bigvee_{i=1}^{r} \theta(x,y,z_i) \wedge \eta(x,y) \Big) \right).$$

Then $\epsilon(y) \in tp(b)$ and by (**), if b' satisfies $\epsilon(y)$ then $\eta(x, b')$ has Morley rank n and Morley degree at most kr. This completes the proof of $(a)_n$.

Proof of $(b)_n$. Let us work inside a sort ($\rho(x)$ or a partial type) S of Morley rank $n + 1$ which we may assume to be of Morley degree 1. Let x be a variable ranging over this sort. Let $p_0(x)$ be the unique (stationary) complete type (over the base parameters $= \emptyset$ say) of Morley rank $n + 1$. Fix a formula $\phi(x, y)$. By the definability of $p_0(x)$ there is $\psi_0(y)$ over $\emptyset$ such that for any b', $\phi(x, b')$ has Morley rank $n + 1$ iff $\psi_0(b')$. Moreover in this case $\phi(x, b')$ has Morley degree 1 too. For each complete $p(x, y)$ containing $\phi(x, y)$ such that for some b, $p(x, b)$ is consistent and of Morley rank $n_p \leq n$, let $\phi_p(x, y)$, $\psi_p(y)$ and k_p be as given by $(a)_{n_p}$ such that $\phi_p(x, y) \rightarrow \phi(x, y)$. By compactness some finite disjunction of ψ_0 and the ψ_p's cover y-space. Everything follows easily. The lemma is proved. □

Now Proposition B.1 follows easily:

Proof of Proposition B.1. By remark B.2 (i) and Lemma B.3 (a), we obtain (i) of the Proposition. And (ii) follows from the lemma by compactness, as in the proof of $(b)_n$ from $(a)_n$. □

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