Calculus: An intuitive approach

Wai Yai Pong

February 5, 2018

Contents

Part I

Single Variable

Chapter 1

Differentiation

1.1 Motivation

We begin with the velocity-time graph of a smooth ride.

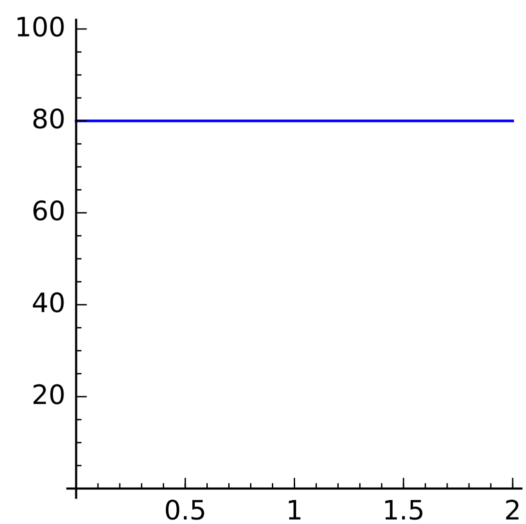


Figure 1.1: The velocity-time graph of a car.

It was boring for the driver since the car was traveling at a constant speed. In other words, the *acceleration*, i.e. the *rate of change of velocity* is zero at any time of this trip. Note that this fact is reflected by the slope of the graph (a horizontal line in this case) being zero at every point. What else about the trip can we tell from this graph? Well, the driver traveled 2 hours at the constant speed of 80 mph, so she traveled $80 \times 2 = 160$ miles. But can we "see" this information from the graph? Yes, it is the area of the rectangle underneath the graph!

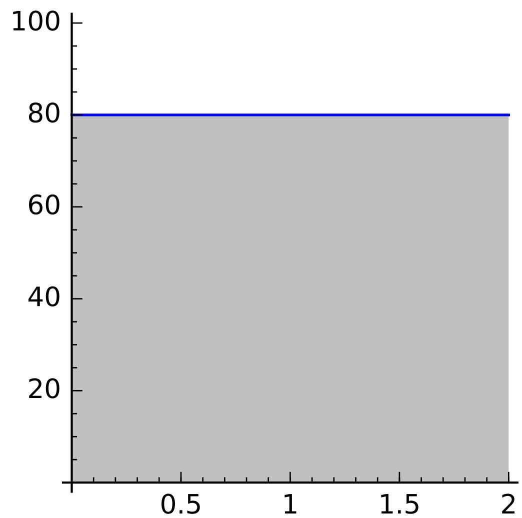


Figure 1.2: The shaded area represent the distance traveled by the car.

However, as you can expect the velocity-time graph of a trip in real traffic is much more complicate. In fact, even nice functions in mathematics, polynomials, trigonometric functions, exponential functions, etc, are still far more complex that a constant function... We surely need some tools to analyze them!

1.2 Derivative

In the previous section we have seen some reasons why one may be interested in finding (the slopes of) tangents. We now discuss how to find them. First, a working definition:

> The **derivative of a function** f **at a point** a **in the domain of** f is the slope (if exists) of the tangent to the graph f at the point $(a, f(a))$.

This definition assumes we know what a tangent is. In fact, it assumes a function, or rather its graph, can have at most one tangent at each point. We will give a precise definition of derivative in Section 1.6. For now let us think of the tangent of a graph at a point p as what the secant $\overline{pq}$ turns into when q approaches to p. We often refer the tangent of the graph of f at the point $(a, f(a))$ simply as the *tangent of f at a*.

A geogebra applet showing how the secants turn into the tangent.

The two most common notation for the derivative of f at a are

$$f'(a) \quad \text{and} \quad \frac{df}{dx}(a).$$

The first one is due to Lagrange, besides being compact, it suggests that the derivatives of f at various points are the values of another function namely f'. The second notation is due to Leibniz. It not only reminds us that a derivative is related to some sort of ratio but also reminds us that f is being considered as a function of x. So, for example, if f is considered as a function of z, then the second notation changes to $\dfrac{df}{dz}(a)$. If we think of a function f as how a dependent valuable y is relating to an independent valuable x, then we may also write its derivative at a as

$$\frac{dy}{dx}(a), \quad \left.\frac{dy}{dx}\right|_{x=a}, \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{(a,f(a))}.$$

Definition 1.2.1. We say that f is **differentiable** at a if $f'(a)$ exists and say that f is **differentiable** on a set D if f is differentiable at a for every $a \in D$. The function $a \mapsto f'(a)$ is called the **derivative** of f. We denote the derivative of $f(x)$ as $f'(x)$ or $\dfrac{df}{dx}$.

Intuitively, it is also clear that differentiability is a *local property*: that means if f and g agree near a, i.e. on an open interval containing a, then $f'(a) = g'(a)$ if either of them exists.

Example 1.2.2. Suppose $f(x) = mx + b$ is a linear function. The graph of f is the line $y = mx + b$ and the tangent at every point is the line itself. Hence $f'(x)$ is the constant function m. In particular, the derivative of a constant function ($m = 0$) is the zero function and the derivative of the identity function $f(x) = x$ is the constant function 1.

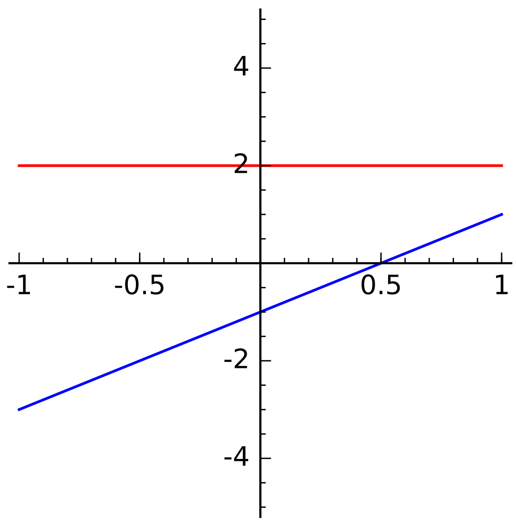


Figure 1.3: The graph of $2x - 1$ (blue) and its derivative (red).

Example 1.2.3. Consider the function $r(x) = 1/x$

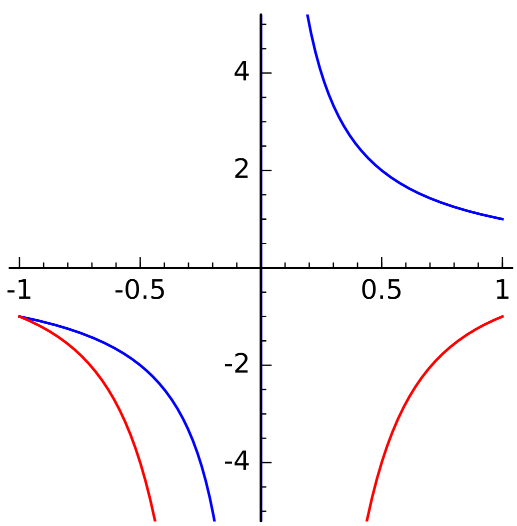


Figure 1.4: The graph of $\frac{1}{x}$ (blue) and its derivative (red)

The slope of the secant of two points on the graph is

$$\frac{r(b) - r(a)}{b - a} = \frac{1/b - 1/a}{b - a} = -\frac{1}{ba}.$$

Letting $b \to a$, we see that the derivative of $1/x$ at the point a is $-1/a^2$. Since $a \neq 0$ is arbitrary, we conclude that $(1/x)' = -1/x^2$. Let us remark that in order for $b \to a$, we need a and b are of the same sign since the domain of $1/x$ does not include 0.

Example 1.2.4. Consider the absolute value function.

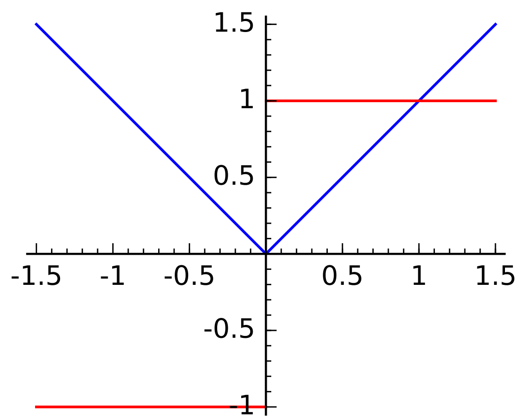


Figure 1.5: The graph of $|x|$ and its derivative

Clearly, the derivative of $|x|$ is 1 at every positive real and -1 at every negative real. The secant joining $(0, 0)$ and $(a, |a|)$ is $y = x$ if $a > 0$ and is $y = -x$ if $a < 0$. They do not approach to the same line when a approaches 0. Therefore, $|x|$ is not differentiable at the origin. This is our first example of a function that is not differentiable at certain point. Roughly speaking, a graph has no tangent at "sharp corners".

Example 1.2.5. Let $f(x) = \sqrt{1 - x^2}$. Even though we do not know $f'(x)$ explicitly yet, nonetheless intuitively $f'(0)$ should be 0 since the corresponding tangent is horizontal. Moreover, f' does not exists at $x = \pm 1$ since the corresponding tangents are vertical hence do not have a slope.

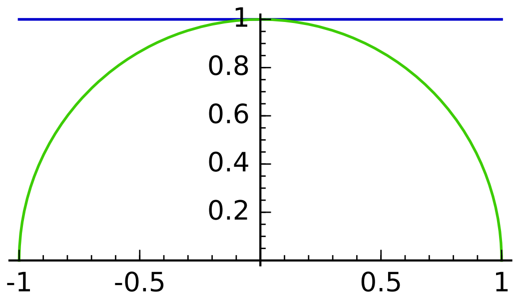


Figure 1.6: The graph of $\sqrt{1-x^2}$ and its tangent at $(0,1)$

1.3 Rules of Differentiation

So far we can only find the derivatives of a very small class of functions. To do more we need some practical ways of computing derivatives.

1.3.1 Sum and Product Rules

Theorem 1.3.1. *Suppose f, g are both differentiable at a then*

1. $(f+g)'(a) = f'(a) + g'(a)$
2. $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

The first statement tells us the derivative of a sum is the sum of the derivatives of the summands. Perhaps no surprise there. The second statement is called the **product rule**. It tells us how to compute the derivative of a product in terms of its factors and their derivatives. This is probably the "first surprise" in calculus since $(fg)'(a) \neq f'(a)g'(a)$ in general[1]. Let us note that if either f or g is not differentiable at a, their sum and product might still be differentiable at a. Let us illustrate this point by a couple examples:

Example 1.3.2. Let $f(x) = |x|$ and $g(x) = -|x|$. As we have seen both f and g fail to be differentiable at 0. However, $(f+g)(x) \equiv 0$ hence differentiable everywhere and at 0 in particular.

Example 1.3.3. Let $f(x) = |x|$ and $g(x) = x$. Then $f(x)$ is not differentiable at 0

[1] But this is a nice surprise. The theory of Calculus would be far less interesting if otherwise.

hence the right-hand side of (2) does not make sense at $x = 0$. However,

$$(fg)(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

thus the secants on either side of 0 approaches to the line $y = 0$ therefore $(fg)'(0) = 0$.

Let us also illustrate how to compute the derivatives of polynomials and power functions using the two rules in Theorem 1.3.1.

Example 1.3.4. Since the derivative of a constant function is the zero function, the product rule implies:

$$(cf(x))' = (c)'f(x) + 2f(x)' = 0(f(x)) + cf'(x) = cf'(x).$$

Thus we can "take the constant out" when taking the derivative.

Example 1.3.5. Since $x' = 1$, it follows from the product rule that

$$(x^2)' = x'x + xx' = x + x = 2x.$$

From this we can deduce

$$(x^3)' = (x^2)'x + x^2(x)' = 2x(x) + x^2(1) = 3x^2.$$

If you are familiar with mathematical induction, it is an easy exercise to show that $(x^n)' = nx^{n-1}$ for any natural number n. Since single variable polynomials can be constructed from the variable and constants via addition and multiplication. We can compute the derivative of polynomials rather easily. For example,

$$\begin{aligned}(2x^3 - x^2 + 5x + 8)' &= (2x^3)' + (-x^2)' + (5x)' + (8)' \\ &= 2(x^3)' - (x^2)' + 5(x)' + (8)' \\ &= 2(3x^2) - 2x + 5 + 0 \\ &= 6x^2 - 2x + 5.\end{aligned}$$

In Example 1.2.3, we show that the $r(x) = 1/x$ is differentiable. Using the product rule and induction, one can show that $(x^n)' = nx^{n-1}$ holds even for negative n. In fact, the equality still holds for any real exponent.

Theorem 1.3.6 (Power Rule)**.** *For any $r \in \mathbb{R}$, $(x^r)' = rx^{r-1}$.*

We will prove the **power rule** in Section 1.6. As an example, take $r = 1/2$ then the power rule asserts that

$$(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Example 1.3.7. Let us compute the derivative of $f(x) = \sqrt{x-2}$ with respect to x. At the first sight, it seems that none of the rules that we have discussed can use to solve the problem. However, we will show that it is not the case and in fact offer two "different" ways of solving the problem. First by examining the graph of f, we see that it is differentiable everywhere except at $x = 2$.

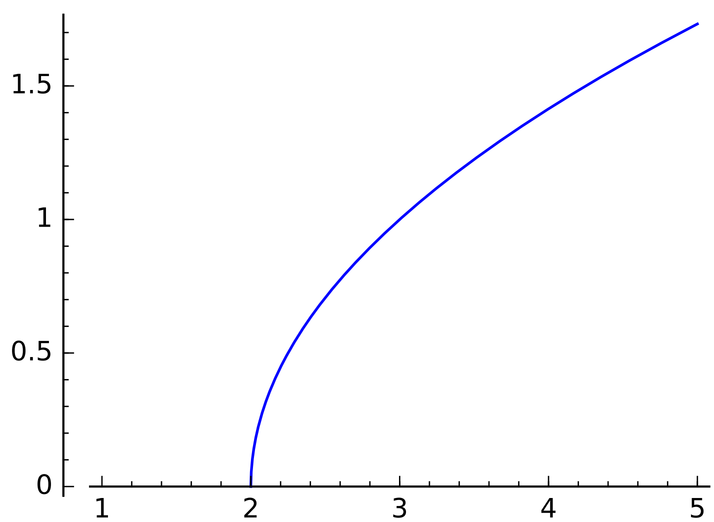


Figure 1.7: The graph of $\sqrt{x-2}$

And so for $x \neq 2$, $f'(x)$ can be computed by differentiating $(f \cdot f)(x) = x+2$ using the product rule. That gives $2f'(x)f(x) = 1$ and so

$$f'(x) = \frac{1}{2f(x)} = \frac{1}{2\sqrt{x-2}}.$$

The second way is more geometrical.

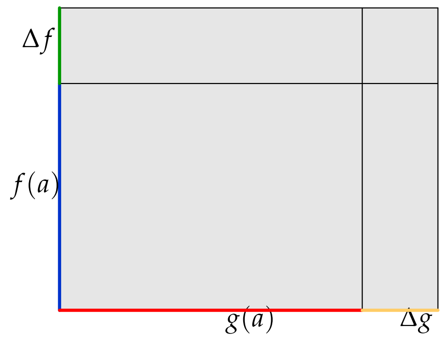


Figure 1.9: A digram for the product rule

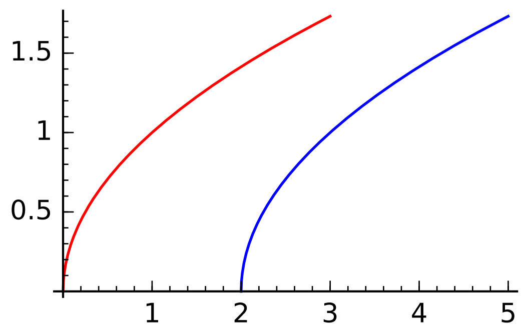


Figure 1.8: The graph of $\sqrt{x}$ and the graph of $\sqrt{x-2}$

Since the graph if $\sqrt{x-2}$ is a horizontal shift of the graph of $\sqrt{x}$, the same relation holds for their derivatives. Hence, the graph of the derivative of $\sqrt{x-2}$ must be just the shift of the graph of $(\sqrt{x})' = 1/(2\sqrt{x})$ (by power rule) to the right by 2. Thus $f'(x)$ must be $1/(2\sqrt{x-2})$.

We shall proof the product rule in Section 1.6. Here we will give an idea why it should take that form by a digram (Figure 1.9). The rectangle represents the quantity $fg + \Delta(fg)$ when x is changed by Δx at $x = a$ [2]. From the figure we see that

$$(fg)(a) + \Delta(fg) = f(a)g(a) + f(a)\Delta g + g(a)\Delta f + \Delta f \Delta g$$
$$\Delta(fg) = f(a)\Delta g + g(a)\Delta f + \Delta f \Delta g.$$

[2] Δfg maybe negative, but the reader should get the idea.

As $\Delta x \to 0$, the left-side of the equation above tends to $(fg)'(a)$ and on the right-side $\Delta f/\Delta x \to f'(a), \Delta g/\Delta x \to g'(a)$ and $(\Delta f)(\Delta g)/\Delta x$ tends $f'(a)(0) = 0 = (0)g'(a)$. Thus, we should expect,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

1.3.2 Exercises

1. Compute the derivative of the following functions.

   (a) $f(x) = \sqrt{3}$.

   (b) $f(x) = \sqrt{x}$.

   (c) $f(x) = 2x^3 - x + 1$.

   (d) $f(x) = \dfrac{x^5 - 3x^2 + 10x}{\sqrt{x}}$. (For this questions, instead of using the quotient rule, try to carry out the division first.)

   (e) $f(x) = 2x^{1/3} - x^{5/7}$.

2. Let $f(x) = 2x^2 + 3x - 1$

   (a) Find the slope of the line tangent to $f(x)$ at $x = -1$.

   (b) Find an equation of the tangent to $y = f(x)$ at $x = -1$.

3. Given $f(x) = x^3 h(x)$ and $h(-1) = 2$, $h'(-1) = 5$. Find $f'(-1)$.

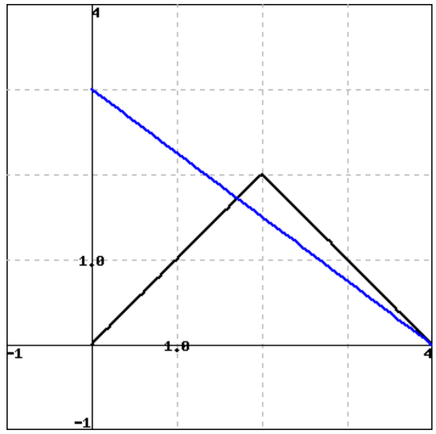
4. Given $f(2) = -1$, $f'(2) = 2$, $g(2) = 2$ and $g'(2) = -2$. Find $h'(2)$ if

   (a) $h(x) = f(x)g(x)$

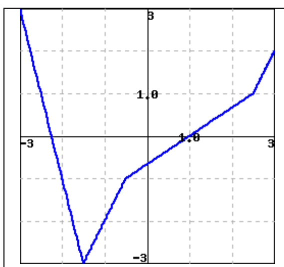
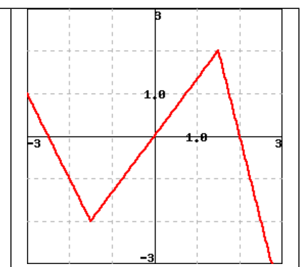
   (b) $h(x) = f(x)/g(x)$

   (c) $h(x) = f^2(x)$.

5. Use the graphs below to find the indicated derivatives. If the derivative does not exist, state so. The graph of $f(x)$ is black and has a sharp corner. The graph of $g(x)$ is blue. Let $h(x) = f(x) \cdot g(x)$. Find $h'(1), h'(2)$ and $h'(3)$.



6. Let $h(x) = f(x) \cdot g(x)$ and $k(x) = f(x)/g(x)$. Use the graphs below to find $h'(2)$ and $k'(-1)$.

	
$f(x)$	$g(x)$

7. Show that $(x^{\frac{1}{q}})' = \frac{1}{q}x^{\frac{1}{q}-1}$ for any integer $q \neq 0$. Deduce that $(x^r)' = rx^{r-1}$ for any real number r.

8. Justify that $\sqrt{x}$ is differentiable for all $x > 0$ by computing the limit $\lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$. (Hint: Rationalize the numerator.)

9. Consider the following two functions defined on $[0,4]$

$$f(x) = \begin{cases} x & 0 \leq x \leq 2 \\ 4-x & 2 \leq x \leq 4 \end{cases}. \qquad g(x) = x - 2.$$

   (a) Graph $f(x)$ and $g(x)$.

   (b) Let $h = fg$. Compute $h'(1)$ and $h'(3)$.

   (c) The function h is indeed differentiable at $x = 2$. Find the value of $h'(2)$. Why does it not contradict the product rule?

1.3.3 The Chain Rule

Even with the rules that we have developed, we can justify the differentiability for only a fairly small class of functions. The situation changes once we established the **Chain Rule**. It deals with differentiability of compositions of functions. It is the key result for justifying differentiability of functions and finding their derivatives.

Example 1.3.8. Before stating the chain rule, let us consider

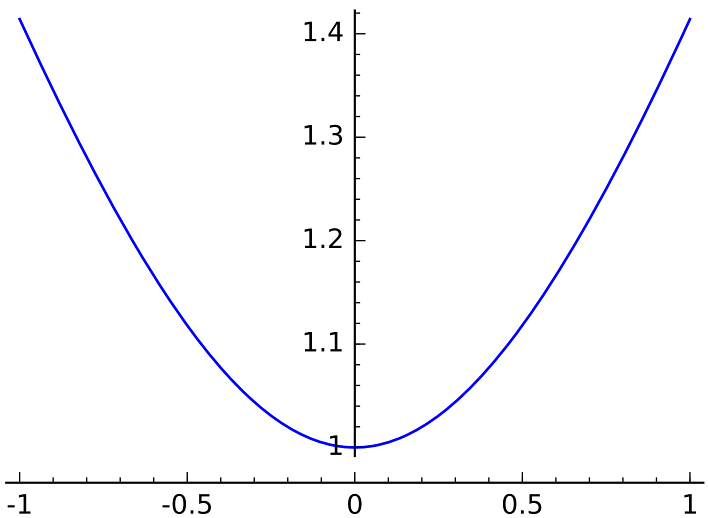


Figure 1.10: The graph of $\sqrt{x^2+1}$

From its graph, it is clear that $h(x) = \sqrt{x^2+1}$ is differentiable (at least on the part that it is shown). However, we cannot justify this fact by the rules that we have so far. Note that $\sqrt{x^2+1}$ is the composition $g \circ f(x)$ where $g(x) = \sqrt{x}$ and $f(x) = x^2 + 1$ both of them we know are differentiable and can compute their derivatives. So what we need is a result about differentiability that deals with composition of functions and that is exactly what the chain rule is about.

On the other hand, if we accept the fact that $h(x)$ is differentiable by the graphical justification, then we do can compute its derivative from the rules that we have so far. This is because $h^2(x) = x^2 + 1$ and so by the product rule $2h(x)h'(x) = 2x$ and hence $h'(x) = x/h(x) = x/\sqrt{x^2+1}$.

Theorem 1.3.9 (Chain Rule)**.** *Suppose f is a function differentiable at a and g is a* *function differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g(f'(a))f'(a) \tag{1.1}$$

We will give a proof of the chain rule in Section 1.6. Just like the previous two rules, the chain rule is not applicable if either $g'(f(a))$ or $f'(a)$ does not exist (see exercises for examples).

For computation, Equation 1.1 is what we need. It expresses the derivative of a composition, in terms of its components and their derivatives.

As an application, let us deduce the **quotient rule** from the product rule and the chain rule. Suppose $g(x)$ is differentiable at a and $g(a) \neq 0$. Since $r(x) = 1/x$ is differentiable (Example 1.2.3). According to the chain rule, the function $(r \circ g)(x) = 1/g(x)$ is differentiable at a, moreover

$$\left(\frac{1}{g}\right)'(a) = (r \circ g)'(a) = r'(g(a))g'(a) = -\frac{g'(a)}{g^2(a)}$$

Now suppose f is also differentiable at a, it then follows from the product rule that

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\left(\frac{1}{g}\right)(a)\\ &= -\frac{f(a)g'(a)}{g^2(a)} + \frac{f'(a)}{g(a)}\\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.\end{aligned}$$

This is the **quotient rule**.

Suppose f is differentiable at a and has an inverse near a. Since the graph of f^{-1} is obtained by reflecting the graph of f along the line $y = x$, intuitively if f has a tangent at $(a, f(a))$, then f^{-1} will have a tangent at $(f(a), a)$. In fact, it would be the line obtained by reflecting the tangent of f at a along the line $y = x$ hence its slope of would be the reciprocal of the slope of the tangent of f at a. Indeed, we have

Theorem 1.3.10. *Suppose f has a nonzero derivative at a and has an inverse near a. Then f^{-1} is differentiable at $f(a)$, moreover,*

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}. \tag{1.2}$$

We will give a proof of this theorem in Section 1.6. The hard part is again to show that f^{-1} is differentiable at $f(a)$. Once this fact is established, then (1.2) is a simple consequence of the chain rule. Since $f^{-1}(f(x)) = x$, differentiate both sides with respect to x and evaluate at a yields,

$$(f^{-1})'(f(a))f'(a) = 1.$$

We obtain (1.2) on dividing both sides by $f'(a) \neq 0$.

1.3.4 Exercises

1. Find the derivatives of the following functions:

   (a) $(x+1)^{95}$.

   (b) $\left(\dfrac{x^2-2}{4}\right)^5$.

   (c) $\sqrt{4x^2-3x+5}$.

   (d) $(-x^2+1)^8(2x^3-1)^5$.

   (e) $\dfrac{7}{(3x^2-4x+1)3}$.

   (f) $\sqrt{2+\sqrt{3x}}$.

2. Given $F(4) = 5, F'(4) = 5, F(5) = 6, F'(5) = 3$ and $G(4) = 5, G'(4) = 6, G(5) = 1, G'(5) = 3$, find each of the following below or indicate that the derivative that cannot be computed from the given information.

   (a) $H'(4)$ if $H(x) = F \circ G(x)$.

   (b) $H'(4)$ if $H(x) = G \circ F(x)$.

3. A table of values is given below:

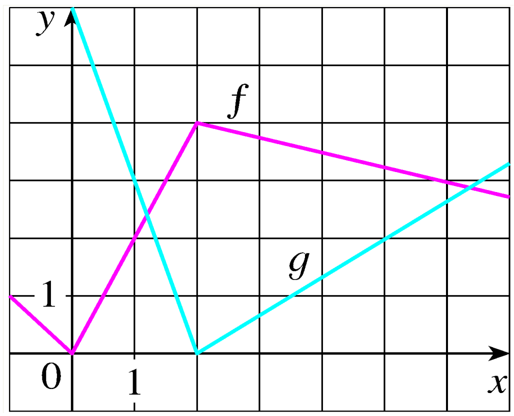
x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

   (a) Fine $F'(2)$ if $F(x) = f \circ f(x)$.

   (b) Find $G'(3)$ if $G(x) = g \circ g(x)$.

4. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x)), v(x) = g(f(x))$, and $w(x) = g(g(x))$. Decide which of the following derivatives

exists and, if exists, compute it.



5. Let $f(x) = \begin{cases} 1+x & x<0; \\ 1+\frac{x}{2} & x \geq 0. \end{cases}$ and $g(x) = \begin{cases} x-1 & x<1; \\ 2x-2 & x \geq 1. \end{cases}$

   (a) Is $f(x)$ differentiable at $x = 0$?

   (b) Is $g(x)$ differentiable at $x = 1$?

   (c) Is $g \circ f(x)$ differentiable at $x = 0$?

6. Let $f(x) = \sqrt{1-x^2}$. Does the derivative of $(f \circ f)(x)$ at $x = 0$ exist?

1.4 Derivatives of Elementary Functions

1.4.1 Trigonometric Functions

To compute the derivatives of trigonometric functions, all we need to know is the derivative of the sine function since other trigonometric functions, e.g. cosine, are just simple transformation of sine. And if you sketch the graph of sine and the graph of its derivative by estimating the slopes of the tangents at various points, you will get a graph that looks very much like the following

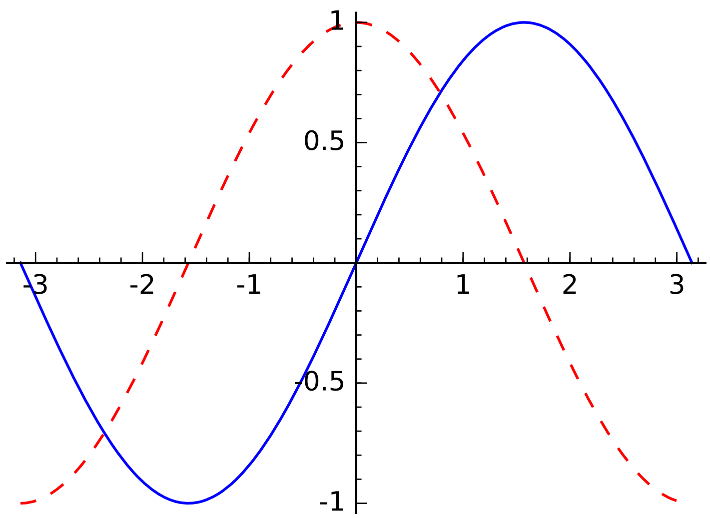


Figure 1.11: The graph of sine (blue) and the sketch of its derivative (red)

So

$$\frac{d}{dx}\sin x = \cos x \tag{1.3}$$

would be a reasonable guess. We will vindicate this intuition in Section 1.6. Since $\cos x$ is the shift of $\sin x$, more precisely, $\cos x \equiv \sin(x + \pi/2)$, the derivative of cosine is the shift, by the same among, of the derivative of sine. That means

$$\cos' x = \cos(x + \pi/2) \equiv -\sin x \tag{1.4}$$

Once we know the derivative of sine and cosine, the derivatives of the other the trigonometric functions follows from the quotient rule.

1.4.2 Exercises

1. Show that $\dfrac{d}{dx}\tan x = \sec^2 x$.
2. Find the derivative of the other trigonometric functions.
3. Find the derivatives of the following functions
   (a) $\sin 2x$.
   (b) $\tan 5x$.
   (c) $\sqrt[3]{1 + \tan x}$.

(d) $-\cot^2(\sin(t))$.

(e) $\cos^4(\sin(3x))$.

(f) $\sin\left(\dfrac{1}{1+x}\right)$.

4. Find the derivatives of the following functions

(a) $f(x) = \sin^3 x$ and $g(x) = \sin(x^3)$.

(b) $f(x) = \cos \dfrac{1}{x}$ and $g(x) = \dfrac{1}{\cos x}$.

5. Find an equation of the tangent to the curve

$$y = \sin(5x) + \cos(2x)$$

at the point $(\pi/6, y(\pi/6))$.

1.4.3 Exponential and Logarithmic Functions

Our goal is to find the derivatives of exponential and logarithmic functions. First, consider an exponential function b^x ($b > 0$). Intuitively, b^x is differentiable everywhere since its graph has no sharp corners and no vertical tangents.

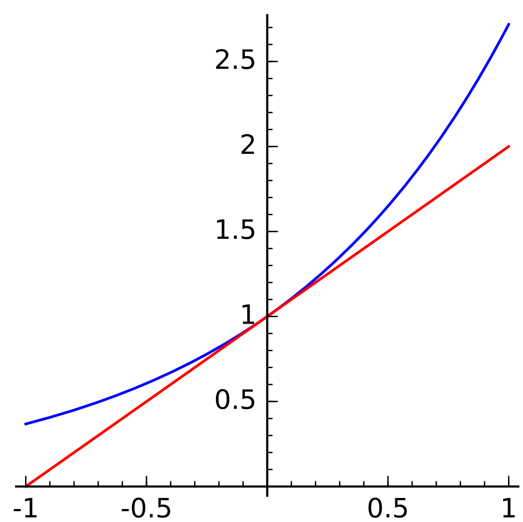


Figure 1.12: The graph of e^x and its tangent at $x = 0$

We will establish this fact in Section 1.6. Meanwhile, note that

$$\frac{b^{x+h} - b^x}{h} = b^x \left(\frac{b^h - 1}{h} \right) \tag{1.5}$$

so by letting $h \to 0$ in (1.5), we conclude that

$$\frac{d}{dx} b^x = b^x \left(\frac{d}{dx} b^x \bigg|_{x=0} \right). \tag{1.6}$$

This reveals an important feature of exponential function:

The derivative of an exponential function is a multiple of itself. Moreover the multiple is precisely the slope of its tangent at $x = 0$.

So it is perfectly natural to ask for which b will make that slope equal 1? In other word, which number b will the derivative of b^x to be itself? From Equation (1.5), we see that it requires $b^h - 1 \approx h$ when $h \approx 0$. In particular, it requires $b^{1/n} \approx 1 + 1/n$, in other words $b \approx (1 + 1/n)^n$, for large n. This prompts us to define the base of this special exponential function to be

$$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n. \tag{1.7}$$

We will show that the limit above exists in Section 1.6 and denote it by e. We call the function $x \mapsto e^x$ (also written as $\exp(x)$) the **exponential function**. Certainly, the whole point of making such a definition is to guarantee

$$\frac{d}{dx} e^x = e^x. \tag{1.8}$$

Let $\ln x$ be the inverse function of e^x (i.e. the logarithm with base e). We call it the **natural logarithm**. According to (1.2) and (1.8), $(\ln x)' = 1/x$. However, to illustrate the proof Theorem 1.3.10 and the advantage of Leibniz notation, we will derive $(\ln x)' = 1/x$ directly. Let $y = \ln x$, so $x = e^y$. Differentiate both sides with respect to x, we get

$$1 = \frac{dx}{dx} = \frac{d}{dx} e^y = \frac{d}{dy} e^y \frac{dy}{dx}$$

according to the chain rule. Thus,

$$\frac{d}{dx} \ln x = \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

1.4.4 Exercises

1. Find the derivatives of the following functions.
   (a) $2x^e + 3e^x$.
   (b) $2\ln(1+x)$.
   (c) $3^x - \dfrac{1}{x^2}$
   (d) $\dfrac{x^3}{3^x}$.
   (e) $\ln\left(\sqrt{(x+1)(x-2)}\right)$.
   (f) $\log_3(x^2+x+1)$.
   (g) 3^{1-x^2}.
   (h) $2^{\sin \pi x}$.
   (i) $e^{x \sin x}$.

2. Find the quadratic polynomial function $g(x) = ax^2 + bx + c$ which best fits the function $f(x) = 2^x$ at $x = 0$ in the sense that $g(0) = f(0), g'(0) = f'(0)$ and $g''(0) = f''(0)$.

3. Assuming that the limit in (1.7) exists, show that $\dfrac{b^h - 1}{h} \to \ln b$ when $h \to 0$ and deduce that
$$\frac{d}{dx}b^x = (\ln b)b^x.$$

4. Use the chain rule to prove again that $(b^x)' = (\ln b)b^x$. (Hint: $b^x = e^{x \ln b}$)

5. Show that $\dfrac{d}{dx}\log_b x = \dfrac{1}{x \ln b}$. (Hint: $\log_b x = (\ln x)/(\ln b)$.)

6. Verify by the chain rule that $(\ln(f))' = \dfrac{f'}{f}$ if both sides exist. (This is called the logarithmic derivative of f.)

7. For functions $f(x)$ and $g(x)$, the symbol $g(x)^{f(x)}$ means the function $e^{f(x) \ln g(x)}$ (also written as $\exp(f(x) \ln g(x))$. The domain of this function is the intersection of the domain of f and the set on which $g > 0$.

   Express the derivative the of g^f in terms of the derivatives of f and g.

8. Find the derivatives of the following functions.
   (a) x^{2x}
   (b) $(x^2+1)^{\ln(x)}$
   (c) $x^{\sin(x)}$.

1.4.5 Inverse Trigonometric Functions

The trigonometric functions are not 1-to-1 so by inverse trigonometric functions we actually mean the inverse functions of the restriction of trigonometric functions to various domains. For example, there are infinitely many numbers whose sine is $1/2$. By $\arcsin 1/2$ we mean the one in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So $\arcsin^{-1}(1/2) = \pi/6$.

Example 1.4.1. Let us compute the derivative of $\arcsin x$. We use the same technique as in computing the derivative of $\ln x$. Let $y = \arcsin x$, then $\sin y = x$ and so

$$1 = \frac{dx}{dx} = \frac{d\sin y}{dx} = \frac{d\sin y}{dy}\frac{dy}{dx} = \cos y \frac{dy}{dx}$$

Thus $dy/dx = 1/\cos y = \sec y$. To express dy/dx back in terms of x, we get a hint from the following diagram

insert a triangle here.

Thus $\sec y = 1/\sqrt{1-x^2}$. Another way to get this is to use the identity

$$1 \equiv \cos^2 y + \sin^2 y = \cos^2 y + x^2,$$

Thus $\cos y = \pm\sqrt{1-x^2}$. By examine the graph of $\arcsin x$, we see that the slopes of its tangents are non-negative hence we should take the positive root.

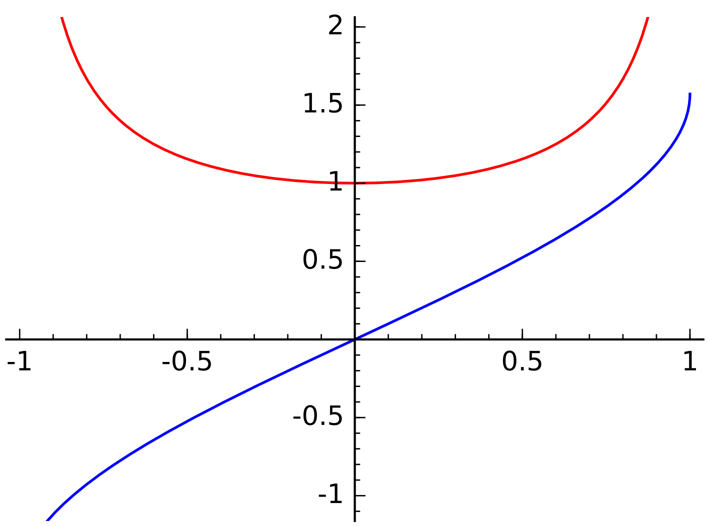


Figure 1.13: $\arcsin x$ and its derivative

In summary, $\arcsin x$ maps $[-1,1]$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$

The graph of cosine restricted to $[0,\pi]$ is a horizontal shift of the graph of sine restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus the graph of $\arccos x$ is a vertical shift of the graph of $-\arcsin x$ and hence the same derivatives (since their tangents at the corresponding points are parallel). Therefore, we conclude that $\arccos x$ maps $[-1,1]$ to $[0,\pi]$ and

$$\frac{d}{dx}\arccos x = -\frac{d}{dx}\arcsin x = -\frac{1}{\sqrt{1-x^2}}.$$

Exercise

1. Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$.
2. Find the derivatives of the other inverse trigonometric functions (pay attention to the domain of each of them).
3. Find the derivative of each of the following functions.
   (a) $x \arcsin(x)$.
   (b) $x^2 \arctan(x)$.
   (c) $\arctan(\cos(4x))$.
   (d) $\operatorname{arcsec}(\sqrt{x^2+1})$.
   (e) $\operatorname{arccsc}(2^x)$.

1.5 Implicit Differentiation

So far we have only studied tangents of curves that are explicitly given as the graph of functions. However, there are curves, for example circles, in the xy-plane to which tangents makes sense but are not graphs of functions of either x or y. In these cases, we can still find the tangents to these curves at various point by treating one of the variable as a function of the other locally. Let $F(x,y)$ be a function of x and y. We say that the relation $F(x,y) = 0$ defines **y implicitly as a function of x near a point** (a,b) if there is a function $y(x)$ defined on an open interval I containing a such that $y(a) = b$ and $F(x, y(x)) = 0$ for all x in I.

Example 1.5.1. The relation $x^2 + y^2 = 25$ defines y as $\sqrt{25 - x^2}$ near $(3, 4)$. The same relation defines y as $-\sqrt{25 - x^2}$ near $(3, -4)$. Let us find the tangent to this circle at the point $(3, -4)$ without explicitly solving y as a function of x. To do this, we differentiate both sides of the equation keeping in mind that y is a function of x.

$$\begin{aligned}
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}25 \\
2x + 2y\frac{dy}{dx} &= 0 \\
x + y\frac{dy}{dx} &= 0.
\end{aligned}$$

So $3 + (-4)\left.\frac{dy}{dx}\right|_{(3,4)} = 0$, therefore $\left.\frac{dy}{dx}\right|_{(3,4)} = \frac{3}{4}$. Thus the tangent to the circle at $(3, -4)$ is $3x - 4y = 25$.

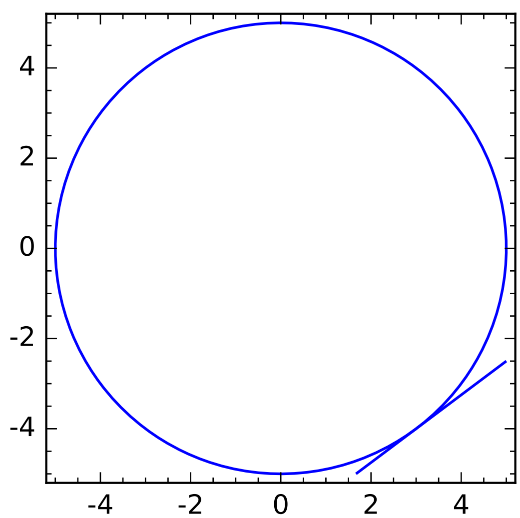


Figure 1.14: The circle $x^2 + y^2 = 25$ and its tangent at $(3, -4)$

Example 1.5.2. From the picture, it is clear that the curve $y^2 = x + x^3$ defines x as a function of y near the origin. Although, it is not easy to express x as a function of y explicitly. Note also that near the origin y is not a function of x.

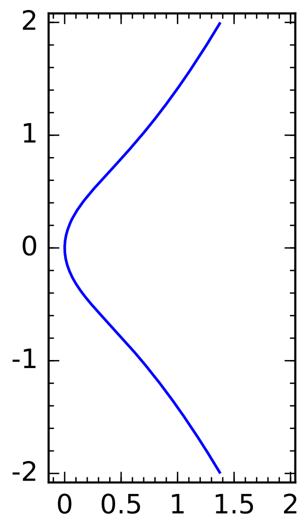


Figure 1.15: The curve $y^2 = x + x^3$

Exercises

1. Find the tangent to each of the following curves at the given points:

   (a) $x^5 + y^2 = 25$ at the point $(0, 5)$.

   (b) $y^2 = \dfrac{x^3}{xy + 6}$ at $(2, 1)$.

   (c) $2\left(x^2 + y^2\right)^2 = 25\left(x^2 - y^2\right)$ at $(3, -1)$.

   (d) $2xy^3 + xy = 15$ at $(5, 1)$.

   (e) $x^2 + y^2 = \left(2x^2 + wy^2 - x\right)^2$ at $(0, 1/2)$.

2. Express the slope of the tangent in terms of x and y to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a, b > 0)$ at (x, y).

3. Consider the ellipse $x^2 + xy + y^2 = 8$.

   (a) Find dy/dx.

   (b) It has two horizontal tangents. Find them.

   (c) It has two vertical tangents. Find them.

1.6 Continuity and Limits

In this section, we show how to put the results that we have encountered so far on solid grounds via the concept of continuity. We will give either proofs or references to those "black boxes" that we have being used and some that we are going to use.

1.6.1 Continuous Functions

What means by a function f is *continuous*? Roughly speaking, that means you can draw the graph of f without ever lifting your pen off the paper. But there are a couple problems:

1. It is not very useful in showing that properties of continuous functions (e.g. compositions of continuous functions is continuous).
2. some functions you simply cannot "draw its graph" (try drawing the graph of $x \mapsto \sin(1/x)$ near 0).

So what makes one can draw the graph without lifting one's pen? Well, one moment of reflection should convince you that: *A function f is continuous at a point x_0 in its domain if the value $f(x)$ is near the value $f(x_0)$ whenever x is near x_0.* If we use $A \simeq B$ to denote the A is "near" (or "close to") B then we can rephrase what we have just said symbolically as

$$x \simeq x_0 \implies f(x) \simeq f(x_0) \tag{1.9}$$

In other words, f is continuous at x_0 means the condition $f(x)$ is "closed to" $f(x_0)$ is guaranteed by the condition x is "close to" x_0. So if we can make precisely what means by "nearby" then we can make precise what means by continuous. It is at this point different branches of mathematics takes different approaches. For example, the approach taken in *Non-standard analysis* is by making the concept of *infinitesimal* precise. This is arguably the historic approach of Calculus taken by Newton and Leibniz and was put onto a solid ground of logic by A.Robinson in 1960. In this approach, the implication in (1.9) as $f(x)$ is infinitesimally close to $f(x)$ whenever x is infinitesimally close to x_0 (reference J.Kiesler). In topology, the approach is by abstracting the properties of "being nearby". A function f is continuous at a point x_0 if $f^{-1}(U)$ is a neighborhood of x_0 whenever U is a neighborhood of $f(x_0)$. For us, we will take up the nowadays "standard" metric approach: f is continuous at x_0 if the distance of $f(x)$ and $f(x_0)$ can be *as small as one pleases* provided that the distance of x and x_0 is *small enough*. This leads to:

Definition 1.6.1. A real-valued function f is continuous at a point $x_0 \in \operatorname{dom} f$ if for any $\varepsilon > 0$, there exists a $\delta := \delta(\varepsilon, x_0) > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. A function is **continuous** on a subset S of its domain if it is continuous at every point of S.

Proposition 1.6.2. *Suppose f and g are continuous a a, then $f + g$ and fg are continuous at a also.*

Proposition 1.6.3. *Suppose f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a. Thus a composition of continuous functions is continuous.*

Theorem 1.6.4 (Intermediate Value Theorem)**.** *Suppose f is continuous on $[a, b]$ and $f(a)f(b) \leq 0$ then $f(c) = 0$ for some $c \in [a, b]$.*

Corollary 1.6.5. *Suppose f is continuous on $[a, b]$ and d is a real number between $f(a)$ and $f(b)$ then there is a $c \in [a, b]$ such that $f(c) = d$.*

Proof. Apply Theorem 1.6.4 to the continuous function $g(x) := f(x) - d$. ☐

1.6.2 Differentiable functions

We adopt Carathéodory's formulation of the derivative.

Definition 1.6.6. Suppose f is a function defined on $X \subseteq \mathbb{R}$ and a is an interior point of X. We say that f is **differentiable** at a, if there exists a function φ continuous at a such that for every $x \in X$,

$$f(x) - f(a) = \varphi(x)(x - a).$$

Moreover, the value $\varphi(a)$ is called the **derivative** of $f(x)$ at a.

We give a proof of the product rule.

Proof. Suppose f and g are differentiable at $x = a$, that mean there exist functions φ, ξ continuous at a such that $f(x) - f(a) = \varphi(x)(x - a)$ and $g(x) - g(a) = \xi(x)(x - a)$ on some open interval I containing a. So on I,

$$\begin{aligned}
f(x)g(x) - f(a)g(a) &= f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a) \\
&= g(x)(f(x) - f(a)) + f(a)(g(x) - g(a)) \\
&= g(x)\varphi(x)(x - a) + f(a)\xi(x)(x - a) \\
&= (g(x)\varphi(x) + f(a)\xi(x))(x - a).
\end{aligned}$$

Since $g(x), \varphi, \xi$ are all continuous at a, so is $g(x)\varphi(x) + f(a)\xi(x)$. Hence the last equation shows that $(fg)(x)$ is differentiable at $x = a$ with derivative $g(a)\varphi(a) + f(a)\xi(a) = f'(a)g(a) + f(a)g'(a)$. ☐

1.6.3 Trigonometric functions

Let us establish the fact $(\sin x)' = \cos x$. First we need the following limit.

Proposition 1.6.7. $\dfrac{\sin(h)}{h} \to 1$ *as* $h \to 0$.

Proof. For small positive h, $\sin(h) < h < \tan(h)$. Thus

$$\cos(h) < \frac{\sin(h)}{h} < 1.$$

Since $\cos(h) \to 1$ and so $\sin(h)/h \to 1$ as $h \to 0$ from the right. By symmetry a similar argument shows that $\sin(h)/h \to 1$ as $h \to 0$ from the left as well. This completes the proof. $\square$

Theorem 1.6.8. $\dfrac{d\sin(x)}{dx} = \cos(x)$.

Proof. First

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)(\cos(h)-1)}{h} + \frac{\sin(h)}{h}\cos x.$$

We need to show that the expression above tends to $\cos(x)$ as $h \to 0$. We know that $\sin(h)/h \to 1$ by Proposition 1.6.7, so it remains to show that $(\cos(h) - 1)/h \to 0$ as $h \to 0$. To see this, note that as $h \to 0$,

$$\frac{\cos(h)-1}{h} = \frac{\cos^2(h)-1}{h(\cos(h)+1)} = -\frac{\sin(h)}{h}\frac{\sin(h)}{\cos(h)+1} \to 0$$

because $\sin(h)/(\cos(h)+1) \to 0$ and again $\sin(h)/h \to 1$ as $h \to 0$. $\square$

1.6.4 A definition of e

We have argued that the exponential function $x \mapsto e^x = \exp(x)$ is differentiable provided that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists. We define e as this limit. Here we justify its existence. Fix an arbitrary natural number $n \geq 2$, according to the binomial theorem:

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + 1 + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k \\
&= 2 + \sum_{k=2}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \\
&< 2 + \sum_{k=2}^{n} \frac{1}{k!} \leq 2 + \sum_{k=1}^{n} \frac{1}{2^k} < 3.
\end{aligned}$$

Note that the k-th term ($k \geq 2$) of the sum

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

increases as n increases. Moreover, the number of terms in the sum also increases as n increases. Thus $(1+1/n)^n$ is increasing and bounded above by 3. Therefore, the limit exists (and is less than 3).

Chapter 2

The Mean Value Theorem and its Applications

We study some applications of differential calculus that we have developed in the previous chapter.

2.1 The Mean Value Theorem

The **Mean Value Theorem** (MVT) is a key theorem in single valuable calculus.

Theorem 2.1.1 (Mean Value Theorem)**.** *Suppose f is a function continuous on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

In other words, if f is continuous on $[a, b]$ and differentiable on (a, b), then the tangent of f at some point $c \in (a, b)$ is parallel to the secant of f connecting the two end points of its graph.

Example 2.1.2. Let $f(x)$ be the restriction of $|x|$ to $[-1, 1]$. Then $f(-1) = 1 = f(1)$. However, the derivative of f is either 1 or -1, so there is no $c \in (-1, 1)$ such that $f'(c) = 0$. This shows that the assumption that the function is differentiable on (a, b) in the MVT is essential.

Example 2.1.3. Consider the function on $[-1, 1]$ defined by

$$f(x) = \begin{cases} x & -1 < x < 1 \\ 0 & x = -1, 1 \end{cases}$$

Clearly $f'(x) = 1$ for all $x \in (-1, 1)$, so there is no $c \in (-1, 1)$ such that

$$f'(c)(1 - (-1)) = 2f'(c) = f(1) - f(-1) = 0.$$

This shows that the assumption that f is continuous on $[a, b]$ in the MVT is also essential.

To proof the mean value theorem, we first establish a special case.

Theorem 2.1.4 (Rolle's Theorem). *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Moreover, if $f(a) = 0 = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. The theorem is trivial if $f \equiv 0$ on $[a, b]$. So suppose f is not identically zero on $[a, b]$. Since $f(a) = 0 = f(b)$, this non-zero extreme value must be attained by some $c \in (a, b)$ (Theorem 2.2.3). Therefore, c must be a local extremum and since $f'(c)$ exists, therefore it must be 0 by the critical point theorem (Theorem 2.2.5). □

Proof of the mean value theorem. Let $s(x)$ be the linear function that defines the secant connecting the two end points of the graph of $f(x)$. Then the difference $g(x) := f(x) - s(x)$ satisfies the assumption of Rolle's Theorem. Therefore, there exists $c \in (a, b)$ such that

$$0 = g'(c) = f'(c) - s'(c) = f'(c) - m$$

where $m = \dfrac{f(b) - f(a)}{b - a}$ is the slope of the secant. □

Definition 2.1.5. Suppose f is defined on an interval I. We say that f is **increasing** (resp. **decreasing**) **on** I if $f(x_1) \leq f(x_2)$ (resp. $f(x_1) \geq f(x_2)$) whenever $x_1, x_2 \in I$ and $x_1 \leq x_2$.

Proposition 2.1.6. *Suppose f is continuous on an interval I and $f' \geq 0$ on the interior of I then f is increasing on I.*

Proof. For any $x_1, x_2 \in I$ with $x_1 < x_2$. Apply the MVT on the restriction of f to $[x_1, x_2]$, we conclude that there is some $c \in [x_1, x_2]$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0,$$

since $f'(c) \geq 0$ by the assumption on f'. □

Applying the argument above to $-f$, we conclude that if f is continuous on I with $f' \leq 0$ on the interior of I, then f is decreasing on I. Therefore, if $f' \equiv 0$ on the interior of I, then f is both increasing and decreasing on I, hence,

Corollary 2.1.7. *Suppose f is differentiable on an interval I and $f' \equiv 0$ on the interior of I. Then f is constant on I.*

More generally,

Corollary 2.1.8. *Suppose f is defined on an open subset of $\mathbb{R}$ and its derivative is constantly 0, then f is locally constant.*

2.1.1 Exercises

1. For each of the following functions, 1) find the slope of the secant line determined by $(a, f(a))$ and $(b, f(b))$ and 2) find all points c in $[a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

   (a) $f(x) = x + \dfrac{4}{x}$, $[a, b] = [1, 8]$.

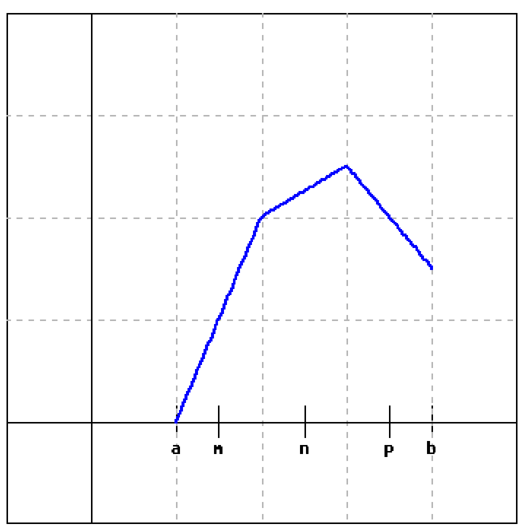
   (b) $f(x) = 3x^2 + 2x + 1$, $[a, b] = [-1, 1]$.

   (c) $f(x) = x\sqrt{x + 3}$, $[a, b] = [-3, 0]$.

2. Suppose $-2 \leq f'(x) \leq 5$ for all $x \in \mathbb{R}$. Give an upper and a lower bound of $f(6)$ if $f(4) = 1$. Hint: first give an upper and a lower bound of the difference $f(6) - f(4)$ using the MVT.

3. Graph $f(x) = 1 - x^{2/3}$ on $[-1, 1]$. Which of the hypotheses of the MVT fails for f?

4. Consider the function f with the following graph



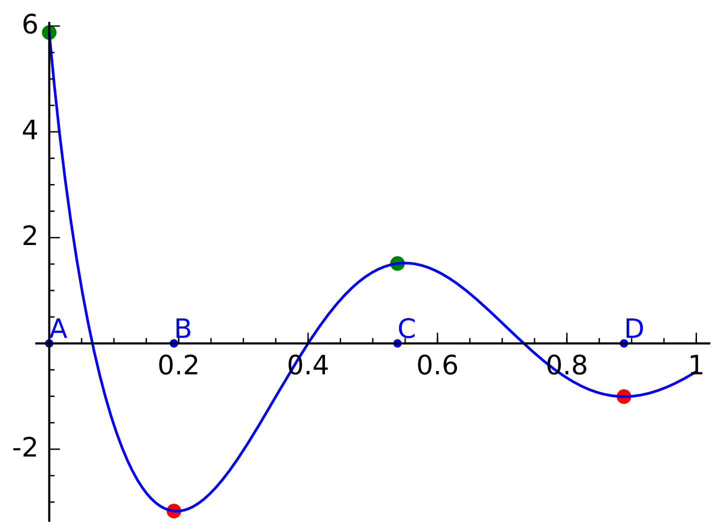
(a) Which of the hypotheses of the MVT fails for f?

(b) Even though f does not satisfy the hypotheses for the MVT, it does not mean that the conclusion is necessarily false. At which of the named pointed is the slope of the tangent = $\dfrac{f(b)-f(a)}{b-a}$?

5. (**Racetrack Principle**) Suppose f and g both define on a and $f'(x) > g'(x)$ (resp. $\geq$) for all $x > a$. Show that $f(x) > g(x)$ (resp. $\geq$) for all $x > a$.

2.2 Minima and Maxima of Functions

There are many situations in which one may be interested in finding maxima and minima of a function. For example, one may want to maximize profits and minimize cost. In this section, we apply differential calculus to the study minima and maxima of functions. To illustrate the relevant concepts, consider the following graph of a function f:



Point $(A, f(A))$ is a maximum point since the value of the function is largest at A. Similarly Point B is a minimum point.

Definition 2.2.1. A point x_0 is a **global maximum** (resp. **global minimum**) of a function f if $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all x in the domain of f.

In other words, a global maximum (minimum) of f is a point in its domain where f attains its maximum (minimum) value. However, intuitively Point

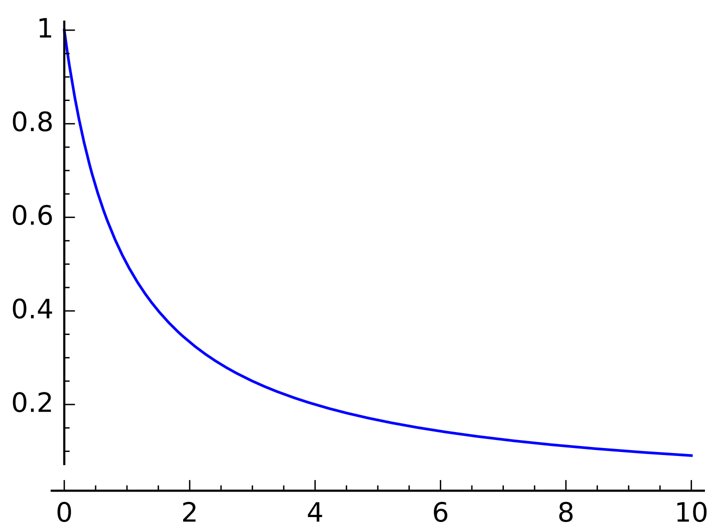
C is also a "maximum" and Point D is a "minimum" even though the function values at these points are neither the largest or the smallest. So in what sense are they extremal points?[1] This prompts us to distinguish another pair of concepts:

Definition 2.2.2. A point x_0 is a **local maximum** (resp. **local minimum**) of a function f if there is an open interval I of center x_0 contained in the domain of f such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in I$.

In the example above, point A is a global maximum of f, while point C is a local maximum and point D is a local minimum. Point B is both a local and a global maximum of the function. Note that the definition of local extremum requires it to be an interior point of the function's domain. Thus a boundary point of the domain, e.g. Point A, cannot be a local extremum.

Certainly, a function needs not have either a maximum or a minimum in its domain as the following examples shows.

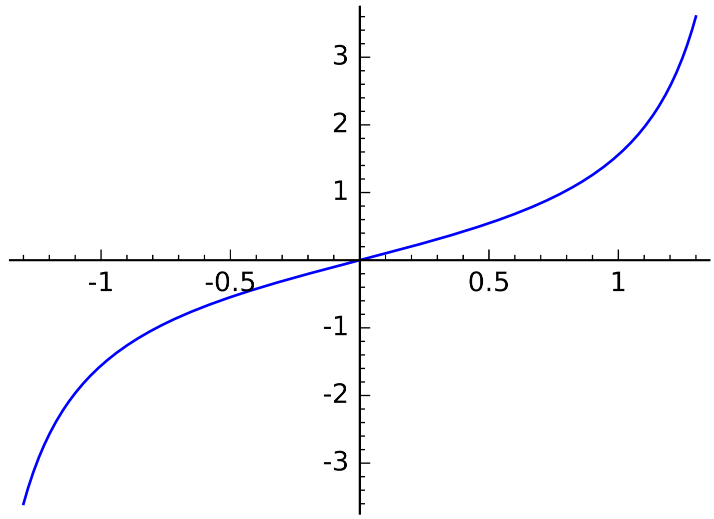
- The function $1/(1+x)$ on $[0, \infty)$ have 1 as its maximum value and is attained at $x = 0$. But it has no minimum value.



- The function $\tan(x)$ on $(-\pi/2, \pi/2)$ has neither global maxima nor global

[1]The word "extremum" means "either maximum or minimum".

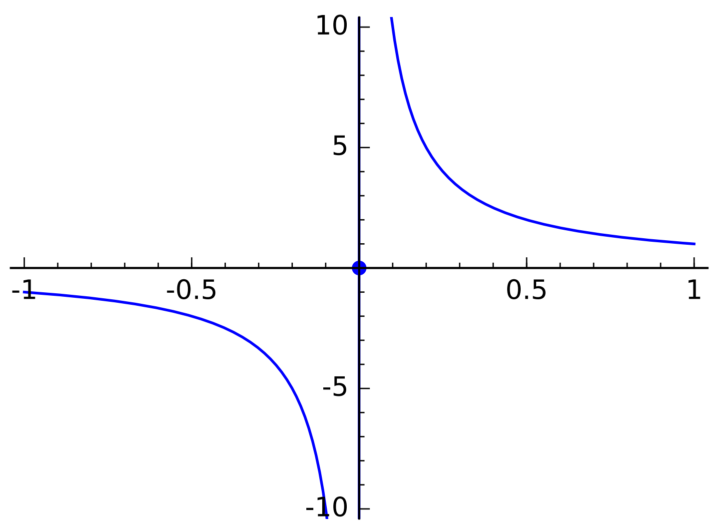
minima.



- The function f on $[-1, 1]$ defined by

$$f(x) = \begin{cases} 1/x & x \neq 0; \\ 0 & x = 0. \end{cases}$$

has neither global maxima nor global minima.



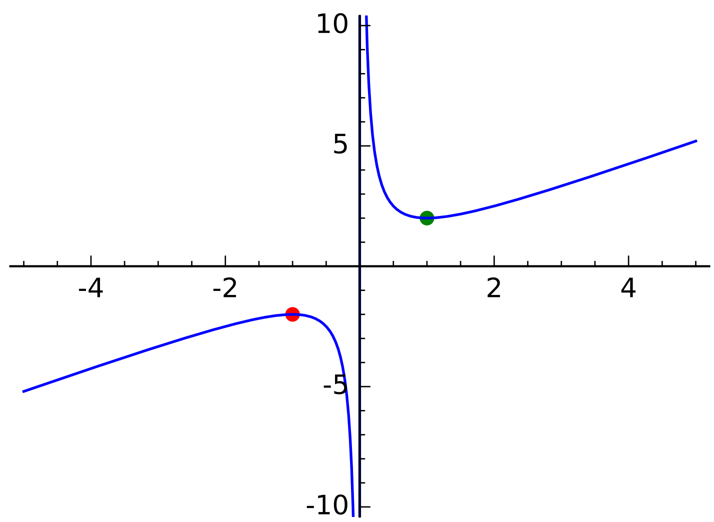
There is, however, a useful and important criterion that guarantees the existence of maximum and minimum.

Theorem 2.2.3 (Extreme Value Theorem)**.** *A continuous function on a closed bounded interval $[a, b]$ has global maximum and global minimum.*

The three examples that come before the theorem show that the assumptions on the Extreme Value Theorem (EVT), i.e. that f is continuous (on its domain) and its domain is a closed bounded interval, are both essential.

The global maximum (minimum) value of f, if exists, is unique however it can be attained at several points in the domain. For example, the absolute value function on $[-1, 1]$ attains its maximum value at $x = \pm 1$. Another example is the cosine, it attains its maximum value at integral multiples of π.

The global maximum value of a function clearly cannot be less than its global minimum value. However, it needs not be the case for its local extremal values as the following graph (of the function $f(x) = x + 1/x$) illustrates.



The value of the function at the green point (a local minimum) is greater than its value at the red point (a local maximum).

So how to find the local maxima and local minima of a function? Intuitively, local extrema are points with horizontal tangents but as we have seen (e.g. $|x|$) they can also be points where the function is not differentiable. This prompts us to make the following definition.

Definition 2.2.4. A point c in the domain of f is a **critical point** of f if either $f'(c) = 0$ or f is not differential at c.

The following theorem justifies our intuition.

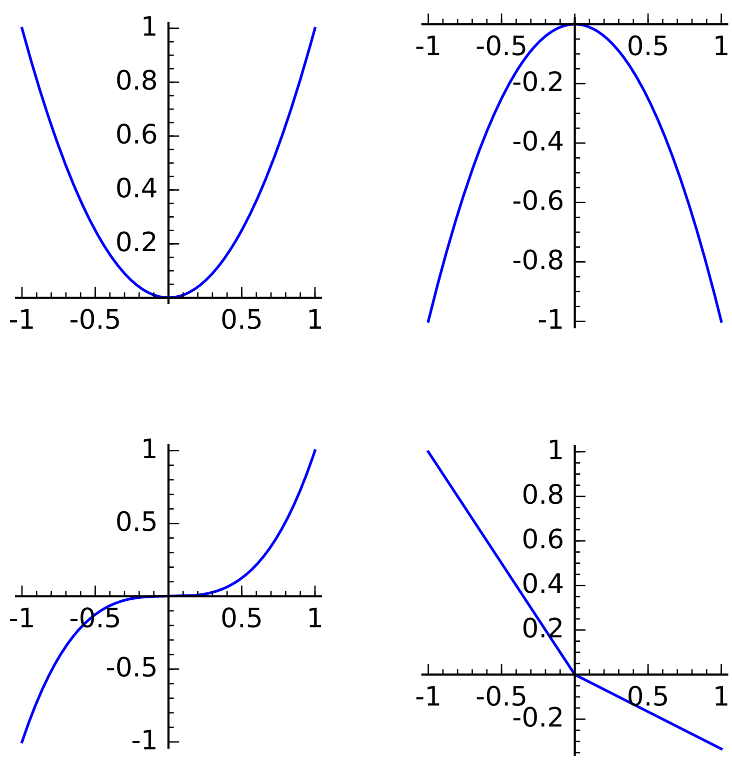
Theorem 2.2.5 (Critical Point Theorem)**.** *Local extrema of f happens only at critical points of f.*

Proof. Without loss of generality, we can assume c is a local minimum (if c is a local maximum of f, then apply the following argument to $-f$). If f is differentiable at c, then on a neighborhood of c there is some function φ continuous at c such that

$$f(x) - f(c) = \varphi(x)(x - c). \tag{2.1}$$

The left-hand side of (2.1) is non-negative on a neighborhood of c, therefore $\varphi(x)$ and $x - c$ have the same sign near c and so $f'(c)$ which is $\varphi(c)$ must be 0 by continuity of $\varphi(x)$ at $x = c$. $\square$

Thus to locate the local extrema of a function, we only need to focus on its critical points. Note, however, that it does not mean that critical points are local extrema. As the following graphs demonstrates.



The origin is a critical point for all four functions. It is a local minimum for the first function, a local maximum for the second but neither a local maximum nor a local minimum for the third and the fourth. So what tells the type of a critical point? It turns to the be the sign of the derivative around the critical point.

Proposition 2.2.6 (First derivative test)**.** *Suppose f is continuous on an open interval (a, b) and $c \in (a, b)$ is a critical point of f. If*

1. *$f' \geq 0$ on (a, c) and $f' \leq 0$ on (c, b) then c is a local maximum of f.*

2. *$f' \leq 0$ on (a, c) and $f' \geq 0$ on (c, b) then c is a local minimum of f.*

If f' does not change sign across c, then c is not a local extremum of f.

Proof. In this first case, by Proposition 2.1.6, f is increasing on (a, c) and is decreasing on (c, b). Therefore, $f(c) \geq f(x)$ for all $x \in (a, b)$. The proof of the second case is similar. □

Since a global extremum that happens at an interior point of the domain of f must also be a local extremum (of the same type), thus we can find them as follows:

1. Find the critical points of f (this includes the boundary points of its domains).

2. Compute the values of f on its critical points, those have the maximum value are the global maxima. And those have the minimum value are the local minima.

Exercises

1. For each of the following functions f, (a) find the open intervals on which f in increasing, (b) find the open intervals on which f in decreasing, (c) find the relative maxima of f, and (d) find the relative minima of f.

   (i) $x^3 - 12x^2 + 45x - 10$.

   (ii) $\dfrac{4x^2}{x} - 6$.

   (iii) $8x^2 - 2x^4$.

   (iv) $\sqrt{x} - x$.

   (v) $1 - \dfrac{3}{x} + \dfrac{3}{x^2}$.

2. For each of the following functions f find the absolute maximum and the absolute minimum values and where do they occur.

   (a) $x^3 + 6x^2 - 63x + 7$ on $[-8, 0]$.

   (b) same function as above but on the interval $[-5, 4]$.

   (c) same function as above but on the interval $[-8, 4]$.

3. Find the absolute maxima and absolute maxima and their corresponding values for each of the following functions on the given interval. If it does not exist, state so.

   (a) $3x^{2/3} - 2x$ on $[-1, 1]$.

   (b) $\dfrac{x}{x^2+1}$ on $[-4, 0]$.

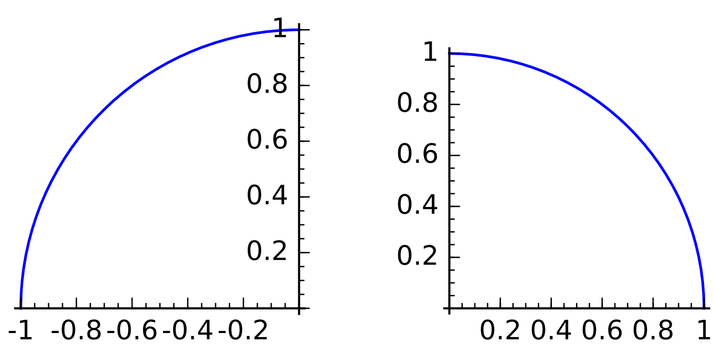
   (c) xe^{-2x} on $(0, \infty)$.

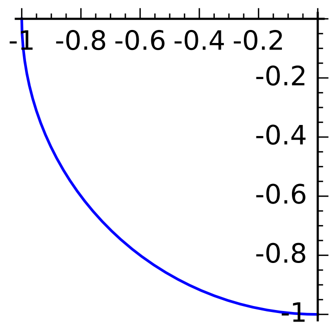
   (d) $\dfrac{x}{1+x^2}$ on $\mathbb{R}$.

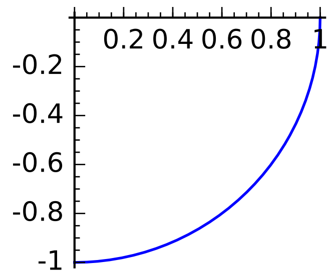
2.3 Concavity

Concavity of a function is a way to tell how its graph curves.

A real-valued function f is **concave up** on an open interval I if its graph is always below the secant over any two points in I. In other words, for any $a, b \in I$ with $a < b$, and any $c \in (a, b)$, $f(c) \leq s(c)$ where s is the secant joining the points $(a, f(a))$ and $(b, f(b))$. A real-valued function f is **concave down** on I if $-f$ is concave up on I.







The upper left picture depicts the graph of an increasing function that is concave down. The upper right picture is the graph of a decreasing function that is concave down. The functions in the bottom row are both concave up, the left one is decreasing and the right one is increasing. These graphs show that monotonicity and concavity are two independent concepts.

The definition of concavity does not rely on the concept of differentiability. However,

Proposition 2.3.1. *If f is differentiable on an open interval I then f is concave up (down) on I if and only if f' is increasing (decreasing) on I.*

Since f'' is the derivative of f', it follows Proposition 2.1.6 that

Proposition 2.3.2. *If f is a twice differentiable function on an open interval I, then f is concave up (down) if and only if $f'' \geq 0$ ($f'' \leq 0$) on I.*

Example 2.3.3. The graph of $f(x) = x^3$ is concave up on $x \geq 0$ and concave down on $x \leq 0$. This is confirmed by Proposition 2.3.1 since $f'(x) = 3x^2$ is increasing on $x \geq 0$ and decreasing on $x \leq 0$.

Definition 2.3.4. A point $x_0 \in I$ (or rather a point $(x_0, f(x_0))$ on the graph of f) is an **inflection point** of f if the concavity of f on either side of x_0 are different.

It follows from Proposition 2.3.1 that an inflection point c of f is a local extremum of f' if $f'(c)$ exists. So c must be a critical point of f' according to the Critical Point Theorem (Theorem 2.2.5).

Example 2.3.5. As we have seen Example 2.3.3, $x = 0$ is an inflection point of $f(x) = x^3$. Note also that, as predicted by the discussion above, $x = 0$ is a critical point of $f'(x) = 3x^2$ because $f''(x) = 6x$ vanishes at $x = 0$.

By flipping the graph along the line $y = x$, we see that $x = 0$ is also an inflection point of $g(x) = \sqrt[3]{x}$. However, 0 is not a critical point of $g'(x) = (1/3)x^{-2/3}$ since g is not even differentiable at $x = 0$.

It follows from the First Derivative Test that

Proposition 2.3.6 (Second Derivative Test)**.** *If $f'' \geq 0$ (resp. $f'' \leq 0$) on an open interval I and c is a critical point of f on I, then c is a global minimum (global maximum) of f on I.*

Note that the assumption in the 2nd derivative test implies $f'(c)$ exists and is 0. We end this section with an interesting result that will be proved later.

Proposition 2.3.7. *If f is both differentiable and concave up (or down) on I, then f' is continuous on I.*

2.3.1 Exercises

1. For each of the following function, (a) find the intervals on which the function is increasing, (b) find the intervals on which the function is decreasing, (c) find the intervals on which the function is concave up, (d) find the intervals on which the function is concave down, and then (e) sketch the graph of the function.

   (i) $x + \dfrac{1}{x}$.

   (ii) $\dfrac{2x+7}{2x+1}$.

   (iii) $x^2 e^x$.

   (iv) $\dfrac{x^3}{x^2-1}$.

   (v) $\dfrac{e^x}{1+e^x}$.

2. For $x_1, x_2 \in I$, let $m(x_1, x_2)$ be the slope of the secant of the graph of f with end points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Show that f is concave up on I if and only if the function $m(x_1, x_2)$ is increasing in x_1 (resp. x_2) for any fixed x_2 (resp. x_1) in I.

3. Formula a similar equivalent condition for concave down.

Chapter 3

Integration

3.1 Anti-derivatives

So far we have been concerning ourselves with the following question: given a function $f(x)$, how to find its derivative $f'(x)$? In this section, we ask the "inverse" problem, namely given a function $f(x)$ find a function $F(x)$ whose derivative is $f(x)$. We call such a function $F(x)$ an **anti-derivative** of $f(x)$. In other words, $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$. The first thing to note is that, unlike its derivative, a function can have many anti-derivatives. For example, x^2 and $x^2 + 1$ are two anti-derivatives of $2x$. We use the notation

$$\int f(x)\, dx$$

to denote the class of all anti-derivatives of $f(x)$. This class of functions is also called the **indefinite integral** of $f(x)$ with respect to x. The second thing to note is that if $F_1(x)$ and $F_2(x)$ are two anti-derivatives of $f(x)$, then they differ by a function whose derivative vanishes (on the domain of f). This is because $(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$. They are, according to the mean value theorem 2.1.1, precisely the **locally constant functions**. In particular, if the domain of f is connected then these are simply the constant functions (identified with the real numbers). It is customary to denote the class of locally constant functions (with the domain understood) by C. Thus we write

$$\int 2x\, dx = \{x^2 + k \colon k \in \mathbb{R}\} = x^2 + C.$$

Quite often C is also referred to as the **integration constant**. Although, as we have seen, it is the class of locally constant functions.

Example 3.1.1. Let us find the indefinite integral of $1/x$. We know that the derivative of $\ln x$ is $1/x$, so $\int 1/x\, dx$ should be $\ln x + C$. However, is it not completely correct. Care must be paid to the domains of these functions. When we say that $(\ln x)' = 1/x$, it is understood that the domain is $x > 0$ since it is the domain of $\ln x$. However, when we start with $1/x$, it is understood that the domain is $x \neq 0$. So $\ln x$ is only "half" of the anti-derivative of $1/x$ (graph them and you will see). So what is the other half? By symmetry, it is not hard to see that the derivative of $\ln(-x)$ is $1/x$ on $x < 0$. Indeed one can check this by differentiation. Thus $\ln|x|$ is an anti-derivative of $1/x$. Hence

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

Also, since the domain $x \neq 0$ has two connected pieces hence the class C of locally constant functions on this domain is the class of functions of the form

$$\begin{cases} c_1 & x > 0 \\ c_2 & x < 0 \end{cases}$$

where c_1, c_2 are, not necessarily the same, constants. For example, the function

$$f(x) = \begin{cases} \ln(x) + 2 & x > 0 \\ \ln(-x) - 3 & x < 0 \end{cases}$$

is an anti-derivative of $1/x$.

3.1.1 Equations of Motion

We discuss a classical application of finding anti-derivative to Physics. The **displacement** of an object with respect to a coordinate system is the position vector of the object. In the 1-dimension case, the displacement of an object is simply its position from the origin. For example, suppose we choose going upward as the positive direction. Then an object -2 feet below ground (here we set the ground at 0) has displacement -2 feet, likewise an object 3 feet above ground has displacement 3 feet.

The motion of an object can be described by $s(t)$ its displacement as a function of time (t). Then $v(t)$ the **velocity** function of the object as a function of t is the rate of change of displacement with respect to t. The **acceleration** of the object $a(t)$, again as a function of time, is the rate of change of its velocity with respect to time. In notation, that means

$$v(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = \frac{dv}{dt}$$

Thus if the acceleration $a(t)$ of an object is known then finding the indefinite integral $\int a(t)\, dt$ tells us what its velocity $v(t)$ looks like since $v(t)$ is an anti-derivative of $a(t)$. In addition, if we know the velocity of the object at a certain time, then we can determine $v(t)$ uniquely. Now since $v(t)$ is found, the same process will give us $s(t)$ if the displacement of the object is know at some point.

Example 3.1.2. A stone thrown upward from the top of a 81 ft cliff at 135 ft/s eventually falls to the beach below.

1. How long does it take for the stone to reach the top?
2. What is the maximum height that the stone reach?
3. How long before the stone hits the beach?
4. What is the velocity of the stone on impact?

To answer these questions, first we choose a coordinate system. A natural choice would be setting the beach at 0, and the upward direction to be the positive direction. In that case, the displacement of the stone initially (at $t = 0$) is 81 feet. The additional information from Physics is that the acceleration due to gravity is -32 ft/s^2. Here the negative sign signifies the velocity is decreasing as time goes by [1]. Since $v(t)$ is an antiderivative of the acceleration, we know that it is a member of the class

$$\int a(t)\, dt = \int (-32)\, dt = -32t + C$$

Hence $v(t) = -32t + k_1$ for some constant k_1. So to find $v(t)$, we need to decide what is k_1. The given information about the initial velocity of the stone: $v = 135$ when $t = 0$, determines k_1 since

$$135 = v(0) = -32(0) + k_1 = k_1.$$

Therefore, we know that $v(t) = -32t + 135$. Since $s(t)$ is an anti-derivative of $v(t)$, so it must be of the form

$$\begin{aligned}\int v(t)\, dt &= \int -32t + 135 = -\frac{32}{2}t^2 + 135t + C \\ &= -16t^2 + 135t + C\end{aligned}$$

[1] In particular when the stone is coming down, it is coming down at a faster speed later. Note the difference between velocity and speed.

Therefore $s(t) = -16t^2 + 135t + k_2$ for some constant k_2. And since initially the stone is 81 feet above the beach (i.e. $s(0) = 81$), therefore

$$81 = s(0) = -16(0)^2 + 135(0) + k_2 = k_2$$

and so $s(t) = -16t^2 + 135t + 81$. Now we can answers the questions. For the first one, note that the stone reaches the top when it stops, i.e. when $v = 0$. So solving $v(t) = 0$,

$$\begin{aligned} 0 &= -32t + 135 \\ t &= \frac{135}{32} \end{aligned}$$

tells us that the stone reaches the top at $t = 135/32 \approx 4.219$ sec. And so the maximum height is $s(135/32) = -16(135/32)^2 + 135(135/2) + 81 \approx 365.8$ feet. Since the beach is $s = 0$, to find the time when the stone hits the beach, we solve $s(t) = 0$,

$$\begin{aligned} -16t^2 + 135t + 81 &= 0 \\ 16t^2 - 135t - 81 &= 0 \\ (t-9)(16t+9) & \end{aligned}$$

and get $t = 9$ or $t = -9/16$. So the stone hits the beach 9 secs after it is thrown upward. Finally, the impacting velocity is $v(9) = -32(9) + 135 = -153$ ft/s.

Exercises

1. Find the indefinite integral of the following functions:

   (a) $x^5 - 2x^3 - x$

   (b) $3x - \sqrt{x}$

   (c) $\dfrac{1}{x^2}$

   (d) $\dfrac{1}{x^3}$

   (e) $2e^x$.

   (f) $2\sin(x)$

   (g) $-\cos(x)$

   (h) $3\sec^2(x) - 3t^2$

A stone thrown upward from the top of a 64 ft cliff at 144 ft/s eventually falls to the beach below. Assume the acceleration due to gravity to be -32 ft/s^2; take upwards to be the positive direction.

(a) How long does the stone take to reach its highest point?

(b) What is its maximum height?

(c) How long before the stone fits the beach?

(d) What is the velocity of the stone on impact?

2. Find $F(x)$ the anti-derivative of $f(x) = \frac{9}{x^2} - \frac{8}{x^6}$ that satisfies $F(1) = 0$.

3. Find the solution of the initial value problem:

$$\frac{dy}{dx} = 1 + \sin(x)$$

such that $y = 5$ when $x = 0$.

3.1.2 Substitution

Sometime it is not immediate what should be the anti-derivate of a given function. For example, $x\sin(x^2)$. Substitution is an idea that help us to recognize the anti-derivative. To facilitate the subsequent discussion, we introduce **differential forms**. We will only treat them on a formal bases as a tool for integration. Readers who want an in-depth treatment of the topic can consult [give a ref]. We think of differential forms instead of functions as the objects to be integrate. So instead of the function $x\sin(x^2)$, we think of the form $x\sin(x^2)\,dx$ as the entity to be integrated. In general, a differential form (in a single variable x) is an expression of the form $f(x)\,dx$. Given a differentiable function $f(x)$, we write $df(x)$ (or simply df) for the differential form $f'(x)\,dx$. We say that a differential form is **exact** if it is $df(x)$ for some differentiable function $f(x)$. Integrating exact form is trivial since by definition $f(x)$ is an anti-derivative of $f'(x)$, so we have

$$\int df(x) = \int f'(x)\,dx = f(x) + C$$

Back to our example, to integrate $x\sin(x^2)\,dx$, first we recognize that

$$\sin(u)\,du = d(-\cos(u))$$

is an exact form. (This is just restating the fact that $-\cos(u)$ is an anti-derivative of $\sin(u)$.) It is reasonable to make the substitution $u = x^2$. So $du = dx^2 = 2xdx$

thus

$$x\sin(x^2)\,dx = \frac{1}{2}\sin(u)\,du = -\frac{1}{2}d\cos(u)$$

and we conclude that

$$\int x\sin(x^2)\,dx = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\sin(x^2) + C.$$

Before giving a few more examples on integration by substitution, we would like to point out that to differentiate $\sin(x^2)$, one would substitute x^2 by a variable and applies the chain rule. In fact, integration by substitution can be viewed as the "anti-chain-rule". As just like the chain rule, one gets better in choosing the right substitution for integration with experience.

Exercises

1. Find the following indefinite integral using substitution.

$$\int x^3\left(1+2x^4\right)^3\,dx$$

   Let $u = 1 + 2x^4$ then

   (a) Find $u'(x)$.
   (b) Using $du = u'(x)dx$ rewrite the indefinite integral as $\int f(u)\,du$.
   (c) Integral out $\int f(u)\,du$ then express the answer back in terms of x.

2. Find the following indefinite integrals:

   (a) $\int x(x^2+4)^2\,dx$
   (b) $\int \frac{1}{e^{2x}}\,dx$
   (c) $\int \cos^3(2x)\sin(2x)\,dx$
   (d) $\int 2x\sin(x^2)\,dx$
   (e) $\int e^{\sin x}\cos x\,dx$
   (f) $\int \frac{e^{2x}}{1+e^{2x}}\,dx$
   (g) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(h) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

(i) $\displaystyle\int \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$

(j) $\displaystyle\int \frac{dx}{x\ln(2x)}$

(k) $\displaystyle\int \frac{\cos x}{2\sin(x)+3}\,dx$

(l) $\displaystyle\int \frac{x^3}{x^4+1}\,dx$

(m) $\displaystyle\int x\sec(2x^2-3)\,dx.$

(n) $\displaystyle\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx$

(o) $\displaystyle\int \frac{2x-1}{(2x^2-2x+1)^3}\,dx$

(p) $\displaystyle\int 3e^{3x}\sin(e^{3x})\,dx$

(q) $\displaystyle\int x^2\sqrt{1+x^3}\,dx$

3.2 Definite Integrals

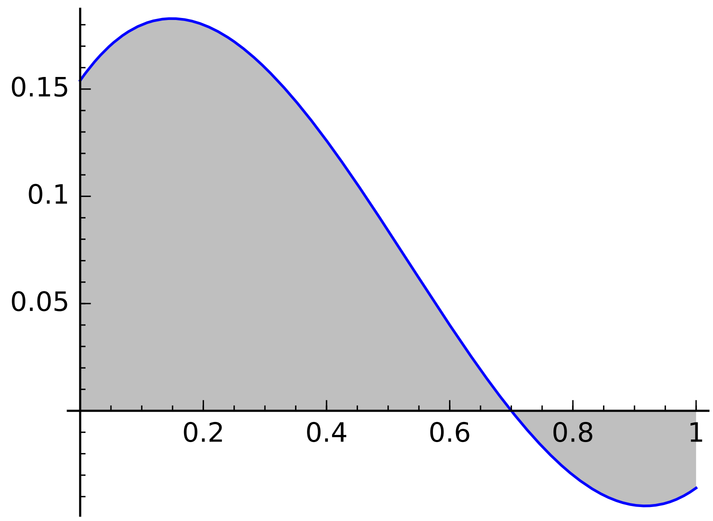
We begin with yet another an imprecise definition: Let $f(x)$ be a function of x defined on an interval $[a,b]$, denoted by

$$\int_a^b f(x)\,dx$$

the *net-area* between the graph of f and the x-axis. We often simply write $\int_a^b f$ if the variable is understood. By net-area here we mean the sum of the areas of the regions above the x-axis minus the sum of the areas of the regions below the x-axis. This definition is vague because it rests on the concept of "area" which is undefined. In fact, once we developed the theory of definite integral carefully, it can be used to give a precise definition of "area". However, our intuition on area will get us some leg in understanding definite integrals. At the moment two things to keep in mind:

1. There are "non-integrable" functions. So roughly speaking that means there are functions to which it does not make sense to talk about the net area between its graph and the x-axis.

2. Continuous functions on a closed bounded interval are integrable.



Moreover, there is a "direction" in the definite integral that is understood: if $a \leq b$, then

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx. \tag{3.1}$$

It follows immediately that for a in the domain of f,

$$\int_a^a f(x)\, dx = -\int_a^a f(x)\, dx \tag{3.2}$$

and hence $\int_a^a f(x)\, dx = 0$. With the net-area interpretation of the definite integral in mind, the following results are intuitive.

Proposition 3.2.1. *If f, g are integrable on $[a,b]$ and $k \in \mathbb{R}$ then $f+g$ and kf are integrable on $[a,b]$. Moreover,*

1. $\int_a^b kf = k\int_a^b f$.
2. $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Proposition 3.2.2. *Suppose f is integrable on $[a,b]$ and $c \in [a,b]$. Then f is integrable on $[a,c]$ and $[c,b]$; moreover*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proposition 3.2.3. *Suppose f is integrable on $[a, b]$ and $f \geq 0$ on $[a, b]$. Then*

$$\int_a^b f \geq 0.$$

Proposition 3.2.4. *If f is integrable on $[a, b]$ and g differs from f at only finitely many points in $[a, b]$, then g is integrable on $[a, b]$; moreover f and g has the same integral over $[a, b]$.*

Exercises

Assume all the functions considered in the following are integrable.

1. Let $A = (-2, 4)$, $B = (1, -3)$, $C = (6, 2)$ and $D = (8, -7)$. The graph of the function $f(x)$ consists of the three line segments AB, BC and CD. Find the integral $\displaystyle\int_{-2}^{8} f(x)\, dx$ by interpreting the integral in terms of sums and/or differences of areas of elementary figures. (Hint: instead of computing the integral directly for f, consider the function g defined by $g(x) = f(x) + 7$. The integral of g over $[-2, 8]$ is easier to find. Then find that for f using properties of definite integrals)

2. Suppose $\displaystyle\int_{-6}^{0} f(x)\, dx = 3$, $\displaystyle\int_{-6}^{-4} f(x)\, dx = 4$, $\displaystyle\int_{-2}^{0} f(x)\, dx = 5$. Find

   (a) $\displaystyle\int_{-4}^{-2} f(x)\, dx$

   (b) $\displaystyle\int_{-2}^{-4} (3f(x) - 4)\, dx$

3. Find the integral $\displaystyle\int_{-5}^{5} \sqrt{25 - x^2}\, dx$ using geometry. Hint: graph $y = \sqrt{25 - x^2}$.

4. Show that $\int_a^b (f - g) = \int_a^b f - \int_a^b g$.

5. Show that if $f \leq g$ on $[a, b]$ then $\int_a^b f \leq \int_a^b g$.

6. (**Integral form of the Mean Value Theorem**) Suppose f is continuous on $[a, b]$. Show that there is a $c \in [a, b]$ such that

$$f(c)(a - b) = \int_a^b f.$$

7. (**Translational invariant**) Suppose f is integrable on $[a, b]$ and $k \in \mathbb{R}$. Show that the function $g(x) = f(x-k)$ is integrable on $[a+k, b+k]$. Moreover,

$$\int_{a+k}^{b+k} g(x)\, dx = \int_a^b f(x)\, dx.$$

8. We say that a subset M of $[a, b]$ is **measurable** if its membership function χ_M is integrable on $[a, b]$. The value $\int_a^b \chi_M$ is the **measure** of M. We say that a property holds **almost everywhere** (a.e.) in $[a, b]$ if the subsets that the property fails is a measure zero set.

   (a) Suppose $f = g$ a.e. on $[a, b]$. Show that f is integrable on $[a, b]$ if and only if g is. Moreover, their integrals on $[a, b]$ are equal.

   (b) Show that every singleton subset of $[a, b]$ has measure 0. Deduce that every finite subset of $[a, b]$ has measure 0.

   (c) Deduce Proposition 3.2.4.

Chapter 4

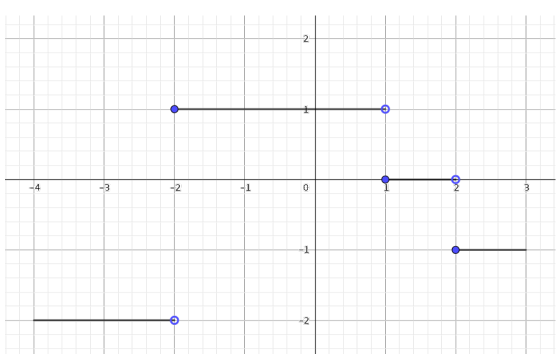
The Fundamental Theorem of Calculus

4.1 Fundamental Theorems of Calculus

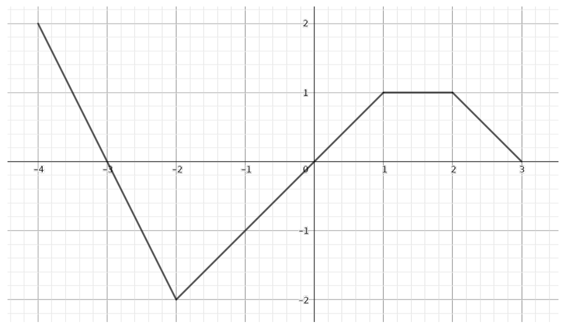
The Fundamental Theorems of Calculus refer to a pair of theorems. Roughly speaking, the pair together say that differentiation and integration are inverse of each other. Before stating them precisely and giving justification, we consider functions given by signed-area. Let f be a function defined on $[a, b]$ and $c \in [a, b]$, The function $F(x)$ on $[a, b]$ defined by

$$F(x) := \int_c^x f(t)\, dt \tag{4.1}$$

measures the signed-area between the graph of f and the x-axis from c. Clearly, the function $F(x)$ depends on the choice of c, so a better notation would be $F_c(x)$. Note that $F(c) = 0$. For example, consider the function f with the following graph:



If we take the lower limit c to be 0, the graph of $F(x) := \int_0^x f(t)\,dt$ is as follows:



If you compare these graphs side-by-side, it is not hard to notice that

1. F is continuous.

2. $F' = f$ whenever F is differentiable (i.e. except at those sharp corners.)

Since F is an integral of f, thus the second point, loosely speaking, is saying that the derivative of an integral is the integrand. On the other hand, if we replace f by F', then again roughly speaking $F = \int f = \int F'$ so the integral of a derivative is the original function. That is also a manifestation of integration and differentiation canceling one another. In fact, it is not hard to make that precise for step functions. And so it is not that far fetch to believe that the same relations hold for any function that can be "well approximated" by step functions. What may be a bit surprising is that the class of functions that can be "well approximated" by step functions is rather "large"; at least it contains most of the functions that we encounter in this course.

We can now state precisely the FTCs and give some justification of them.

Theorem 4.1.1 (FTC 1st form)**.** *Suppose $f, F\colon [a,b] \to \mathbb{R}$ are functions such that:*

1. *F is continuous on $[a,b]$.*

2. *$F'(x) = f(x)$ on $[a,b]$ except possibly at finitely many points.*

3. *f is integrable on $[a,b]$.*

Then

$$\int_a^b f = F(b) - f(a)$$

Sketch of Proof. We simplify by assuming $F' = f$ on $[a,b]$. The general case can then be handled by breaking up the interval into finitely many subintervals so that on each of them $F' = f$. Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a,b]$. Then

$$F(b) - F(a) = \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = \sum_{k=1}^{n} f(u_k)(x_k - x_{k-1}) \tag{4.2}$$

where $u_k \in (x_{k-1}, x_k)$ is guaranteed by the MVT. Note that the sum on the right-side of Equation (4.2) is a Riemann sum of f. By the assumption that f is integrable, it can be as closed to the integral $\int_a^b f$ as one please by choosing P accordingly. However, since the left-side of (4.2) is $F(b) - F(a)$ and is independent of the choice of P. We conclude that $F(b) - F(a) = \int_a^b f$. $\square$

Theorem 4.1.2 (FTC 2nd form). *Suppose f is integrable on $[a,b]$ and $c \in [a,b]$. Let*

$$F(x) = \int_c^x f(t)\,dt.$$

Then F is continuous on $[a,b]$. If f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof. Sketch of Proof Pick a $M > 0$ such that $|f(x)| \leq M$ on $[a,b]$. For any $x, y \in [a,b]$,

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \leq \int_x^y |f(t)|\,dt \leq M(y-x).$$

This shows that F is continuous (in fact uniformly continuous) on $[a,b]$. Suppose f is continuous at $x_0 \in (a,b)$. Note that for $x \in [a,b], x \neq x_0$,

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x [f(t) - f(x_0)]\,dt.$$

Since f is continuous at x_0, the absolute value of the integrand above and hence the right-side of the equation tends to 0 as $x \to x_0$. This shows that F is differentiable at x_0 and then $F'(x_0) = f(x_0)$. $\square$

Exercises

1. Use the 2nd form of the FTC to find the derivative of the following functions:

   (a) $\displaystyle\int_2^x \frac{1}{1+t^2}\,dt$

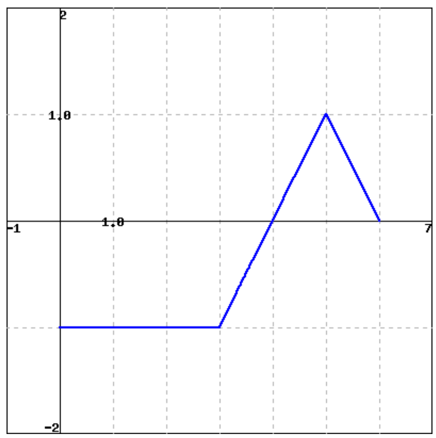
   (b) $\displaystyle\int_x^{x^2} t^3\,dt$

(c) $\displaystyle\int_{2x}^{5x} \frac{t+1}{t-2}\,dt$

(d) $\displaystyle\int_{\sqrt{x}}^{1} \frac{t^2}{1+2t^4}\,dt$

2. Suppose $f(x) = \displaystyle\int_0^x \frac{4-t^2}{1+\cos^2(t)}\,dt$. For what value(s) of x does $f(x)$ have a local maximum?

3. The figure below is the graph of a function $f(x)$.



If F is an anti-derivative of f and $F(0) = -1$. Graph $F(x)$ and find the values of $F(b)$ for $b = 1, 2, 3, 4, 5, 6$.

Chapter 5

Further Results

5.1 Further Results

In this section we will apply what we have learned to obtain a few interesting results.

5.1.1 Composition of Integrable Functions with Continuous Functions

When we discuss integration, we have seen that if g is continuous and f is integrable then $g \circ f$ is integrable. However, it is not true when the composition is taking the other way round.

Example 5.1.1. There is a integrable function f on $[0, 1]$ and a continuous function g on $[0, 1]$ such that $f \circ g$ is not integrable on $[0, 1]$. This example is taken from (with slight modification) the article "Is the Composition Function Integrable" by Jitan Lu. It appeared in the American Mathematical Monthly, Vol. 106, No. 8. (Oct 1999) pp 763-766.

Here is the construction, let f be the function on $[0, 1]$ defined by $f(y) = 1$ for $y \neq 0$ and $f(0) = 0$. Clearly f is integrable since it is monotone (or from the fact that it differs from the constant function 1 at only one point). Next we construct a sequence of continuous function g_n ($n \in \mathbb{N}$) as follows: let $g_0 \equiv 0$. Divides $[0, 1]$ into three subintervals, say I_1, I_2, I_3 of the same length such that I_2 is the middle interval. Let g_1 be the function such that:

1. $g_1 = g_0$ on I_1, I_3.
2. The graph of g_1 together with I_2 form the 3 sides of an isosceles triangle with I_2 as its base and height $1/2$.

Clearly g_1 is continuous. Now suppose g_{n-1} has been defined. Let g_n be the function on $[0, 1]$ with its graph obtained by modifying that of g_{n-1} as follows: Divide each interval on which $g_{n-1} \equiv 0$ (there are 2^{n-1} of them) into three subintervals so that the middle interval has center the same as the original interval and of length $1/3^n \cdot 2^{n-1}$. Modify the graph of g_{n-1} such that each of those middle intervals become the base of an isosceles triangle of height $1/2^n$. Let g_n be the function with the resulting graph as its graph. Clearly g_n is continuous. Moreover for each n, it is clearly from the construction that

$$\sup_{x \in [0,1]} |g_n(x) - g_{n-1}(x)| = \frac{1}{2^n} \to 0$$

as $n \to \infty$. And since the series $\sum 2^{-n}$ converges Therefore, by the Cauchy criteria the sequence (g_n) converges uniformly on $[0,1]$. Let g be the limit function. Since the convergence is uniform and each g_n is continuous, we conclude that g is also continuous on $[0,1]$. Note that g also has the following properties:

1. g is not identically 0 on any subinterval of $[0,1]$.

2. the support of g has measure $1 + 1/3 + 1/3^2 + \cdots = 1/2$.

Finally, let us verify that $f \circ g$ is not integrable on $[0,1]$. Let $P = \{x_k \colon k = 0, \ldots, m\}$ be a partition on $[0,1]$. We divide the Indies k's into two subsets, say A and B. The index k belongs to A if and only if $g > 0$ on the k-th subinterval of P. By (2), $\sum_{k\in A} \Delta_k \le 1/2$ and hence $\sum_{k\in B} \Delta_k \ge 1/2$. But for any $k \in B$, g has both a zero and a non-zero point in the k-subinterval. Therefore $M_k(f \circ g) - m_k(f \circ g) = 1$. And so the different between $U(f \circ g, P)$ $L(f \circ g, P)$ is at least $1/2$. Since the partition P is arbitrarily chosen, therefore we conclude that $f \circ g$ is not integrable.

5.1.2 A no where differentiable continuous function

This classical example (admittedly very strange, and counter intuitive) is due to Weierstrass. The proof given here is taken from Rudin's classic textbook—Principles of Mathematical Analysis. Let's begin with an informal discussion. The absolute value is everyone's first example of a non-differentiable function. Now it only not differentiable at one point. So the idea is to repeat with sharp turn "everywhere". It hard to achieve that in finitely many steps... so we create a sequence of functions with more and more wedges. And hopefully at the limit, we get what we want. As usual, this interesting example cannot be written down by a "formula".

Theorem 5.1.2. *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof. Let $g_0(x)$ is the continuous function on $\mathbb{R}$ defined by requiring its restriction on $[-1,1]$ to be the same as $|x|$ and that g_0 is 2-periodic, i.e. $g_0(x+2) = g_0(x)$ for all $x \in \mathbb{R}$. Note that for any $s, t \in \mathbb{R}$, we have

$$|g_0(s) - g_0(t)| \le |s - t| \tag{*}$$

For each $n \ge 1$, let $g_n(x) = g_0(4^n x)$ and let

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g_n(x).$$

Clearly for each n, $|g_n(x)| \leq 1$ on $\mathbb{R}$ and $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 1/(1-(3/4)) = 4$. Therefore by the Weierstrass' M-test, $\sum \left(\frac{3}{4}\right)^n g_n(x) \to g$ uniformly on $\mathbb{R}$. And so g is continuous on $\mathbb{R}$ since each g_n is.

Next we are going to show that g is nowhere differentiable. Fix any $x \in \mathbb{R}$ and $m > 0$, let $\delta_m = \pm\frac{1}{2\cdot 4^m}$ where the sign of δ_m is chosen so that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. This is doable since the length of the interval with endpoints $4^m(x \pm \delta_m)$ is 1. Define

$$\gamma_n = \frac{g_n(x+\delta_m) - g_n(x)}{\delta_m} = \frac{g_0(4^n(x+\delta_m)) - g_0(4^n x)}{\delta_m}.$$

When $n > m$, $4^n\delta_m$ is an even integer. Since g_0 is 2-periodic, $\gamma_n = 0$. When $0 \leq n \leq m$, the $(*)$ implies, $|\gamma_n| \leq 4^n$. Since $|\gamma_m| = 4^m$, we have

$$\begin{aligned}
\left|\frac{g(x+\delta_m) - g(x)}{\delta_m}\right| &= \left|\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n\right| \\
&\geq 3^m - \sum_{n=0}^{m-1} 3^n \\
&= \frac{1}{2}(3^m + 1).
\end{aligned}$$

As $m \to \infty$, $\delta_m \to 0$, it follows that g is not differentiable at x. □

5.1.3 Integration in Finite Terms

In this section, we discuss integration in finite terms. You have learn all sort of tricks in finding anti-derivatives, for example, substitution, integration by parts, partial fractions, trigonometrical substitution, etc. However, we soon find ourselves "running out-of-tricks". The first frustrating integral that you have encounter is

$$\int e^{-t^2}\, dt.$$

And your teacher might have told you that it cannot be "integrated out". But what means by "integrated out"? Clearly the integrand e^{-t^2} is a continuous function on $\mathbb{R}$ and hence integrable on any closed bounded interval. Therefore the anti-derivative

$$F(x) = \int_0^x e^{-t^2}\, dt.$$

is a perfectly well-defined function on. Why aren't we happy with this? Well, because of the integral sign. Using what we have learned about power series, F is represented by the series obtained by integrating the Taylor series of e^{-t^2} about 0 term-by-term.

$$F(x) = \sum_{k=0}^{\infty}(-1)^k \int_0^x \frac{t^{2k}}{k!}\,dt = \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k+1}}{k!(2k+1)} = t - \frac{t^3}{3} + \frac{t^5}{2\cdot 5} - \frac{t^7}{3!\cdot 7} + \cdots$$

But in same sense, this is still unsettling since the we have to use infinite series (i.e. not in finite terms). Certainly, what is satisfying depends on your point of view of functions. Again, what's wrong with $\int_0^x e^{-t^2}\,dt$? Well, one complaint is that such an expression does not tell us a good way of computing the values of F. In this aspect, the power series expression is far better, since we are get a good approximation of the function value by computing just the first few terms in the series. On the other hand, the integral representation of F is not without merits. For example, it is immediate from the integral expression that $F'(x) = e^{-x^2}$ (FTC) and F is increasing (since the integrand is alway positive). Also that gives F a clear geometric meaning—$F(x) =$ the area under the (bell) curve e^{-t^2} from 0 to x.

Back to the question of integration in terms. Let make precise what do we mean. First

Definition 5.1.3. An algebraic function of a variable x is a root of a polynomial whose coefficients are themselves polynomials in x.

For example, $f(x) = \dfrac{5x^3}{\sqrt{x^2+1}}$ is an algebraic function of x as it is a root of $p(y) = (x^2+1)y^2 - 25x^6$.

Definition 5.1.4. An elementary function of a variable x is a function that can be obtain using x and constants, by a finite number of operations of $+, -, \cdot, \div$, raising to powers, taking roots, forming trigonometric functions and their inverses, taking of exponentials and logarithms.

For example, $\sin x$; $\arcsin x$; $x^x = \exp(x\log x)$; $\tan(\cos^2(x^{4/3}-1) + x^2 - 7)$ are elementary. Now we make the meaning of "f can be integrated out" precise by saying that f has an elementary anti-derivatives. Risch algorithms is an algorithm, based on a long history of works dating back to the early 19th century, for deciding whether an elementary function has an elementary integral. Nowadays most of the computational software has an improved version of the Risch algorithm implemented. For the details, one can consult wikipedia (integration in finite terms) and an article by Elena Anne Marchisotto and Gholam-Ali Zakeri in the College Journal of Mathematics vol 25, No.4, September 2004.

Our first result is due to Laplace,

Theorem 5.1.5 (Laplace(1812))**.** *The integral of a rational function is always an elementary function. In fact, it is either rational or the sum of a rational function and a finite number of constant multiples of logarithms of rational functions.*

We illustrate this by an example

Example 5.1.6.

$$\begin{aligned}\int \frac{(x^2+1)^2+x}{x(x^2+1)}dx &= \int x dx + \int \frac{1}{x} dx + \int \frac{1}{x^2+1} dx \\ &= \frac{1}{2}x^2 + \log|x| + \int \left(\frac{i}{2(x+i)} - \frac{i}{2(x-i)} \right) dx \\ &= \frac{1}{2}x^2 + \log|x| + \frac{i}{2} \log \left| \frac{x+i}{x-i} \right| + C.\end{aligned}$$

The next theorem is due to Liouville, it gives a necessary condition for and algebraic function to have an elementary anti-derivative.

Theorem 5.1.7 (Liouville(1834))**.** *If $f(x)$ is an algebraic function of x and $\int f$ is elementary, then*

$$\int f(x)\, dx = U_0 + \sum_{j=1}^{n} C_j \log(U_j)$$

where the C_j's are constants and the U_j's are algebraic functions of x.

The idea of the proof is that the derivative of an exponential term is exponential, while the derivative of a logarithmic term of order > 1 is logarithmic (trigonometric functions and their inverses can be expressed using exp and log and complex numbers). Thus if the derivative is algebraic and the anti-derivative is elementary then it must of the form on the right-hand-side of the equation above.

Let us illustrate how the trigonometric functions and their inverses can be expressed using exp, log and complex numbers. The key is Euler's famous formula

$$e^{ix} = \cos x + i \sin x.$$

From this, it follows that $\sin x = (e^{ix} - e^{-ix})/2i$ and $\cos x = (e^{ix} + e^{-ix})/2$. Let $x = \sin y (= (e^{iy} - e^{-iy})/2)$ then we get

$$e^{2iy} - 2ixe^{iy} - 1 = 0.$$

Solving the quadratic gives $e^{iy} = ix + \sqrt{1-x^2}$. So

$$\arcsin x = y = i \log(ix + \sqrt{1-x^2}).$$

Liouville generalized this theorem to a much more useful form in 1835 (that form the basis of much of the later works). One special case, of this result is the following theorem:

Theorem 5.1.8 (Special case of Louville 1853 theorem)**.** *If $f(x)$ and $g(x)$ are rational with $g(x)$ non-constant, then $\int f(x)e^{g(x)}$ is elementary if and only if there exists a rational function $R(x)$ such that $f(x) = R'(x) + R(x)g'(x)$.*

According to this form of the Liouville Theorem, to show that $\int e^{-x^2}\,dx$ is not elementary, we need to verify that $1 = R'(x) - 2xR(x)$ has no rational function solutions.

Proof. Suppose not, say $R(x) = p(x)/q(x)$, where p, q are relatively prime, is a solution. Since $R' = (p'q - q'p)/q^2$, we have

$$q^2 = p'q - pq' - 2xpq.$$

This can be written as

$$(q(x) - p'(x) + 2xp(x))q(x) = -p(x)q'(x).$$

So if x_0 is a zero of q of order $k \geq 1$, then x_0 is a zero of order $\geq k$ of the left-hand-side of the equation above. On the other hand, since p and q are relatively prime (hence no common zeros) therefore, x_0 is a zero of order at most $k-1$ of the right-hand-side. That means q has no zero and hence must be a constant. By dividing q, we can assume $q = 1$. And so the equation becomes

$$1 - p'(x) + 2xp = 0$$

which clearly has no polynomial solution (by considering degree). Therefore, $\int e^{-x^2}$ is not elementary. □

Chapter 6

Techniques of Integration

6.1 Integration by Parts

In this section we introduce a technique called **integration by parts**. It takes the form

$$\int u\, dv = uv - \int v\, du. \tag{6.1}$$

Here u and v are the "parts". Equation (6.1) can be regarded as the "anti-product rule" because the product rule, expressed in differential form, looks like

$$d(uv) = udv + vdu. \tag{6.2}$$

Thus taking anti-derivative on both sides yields,

$$\int udv + \int vdu = \int d(uv) = uv + C.$$

By moving $\int vdu$ to the other side, we obtain Equation (6.1). The integration constant is being absorbed into the integral since (6.1) is actually expressing equality of two classes of functions.

Making the variable, say x, explict we have $udv = u(x)dv(x) = u(x)v'(x)dx$. Similarly, $vdu = v(x)u'(x)dx$. So another common way of expressing Equation (6.1) is that

$$\int u(x)v'(x)\, dx = u(x)v(x) - \int v(x)u'(x)\, dx. \tag{6.3}$$

Also, the same kind of argument shows that the relation holds for definite integrals. More precisely,

$$\int_a^b u(x)v'(x)\, dx = u(x)v(x)|_a^b - \int_a^b v(x)u'(x)\, dx. \tag{6.4}$$

Example 6.1.1. We can use integration by parts to find the antiderivative of $\ln(x)$. Take $u = \ln(x)$ and $dv = dx$. Then $du = dx/x$ and we can take v to be x or, as a matter of fact, any antiderivative of x. It follows from Equation (6.1) that

$$\int \ln(x)\, dx = x\ln(x) - \int x\left(\frac{1}{x}\right) dx = x\ln(x) - x + C.$$

Example 6.1.2. Let us use IBP to find the integral $\int x\cos x\, dx$. This time take $u = x$ and $dv = \cos x\, dx$. Then $du = dx$ and v can be taken as $\sin x$. According to (6.1), we have

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx = x\sin x + \cos x + C$$

Sometime we need to apply IBP more than once.

Example 6.1.3. Let us find the integral $\int x^2 \sin x\, dx$. Integration by parts once yields,

$$\begin{aligned}\int x^2\sin x\, dx &= \int -x^2 d\cos x = -\left(x^2\cos x - \int \cos x dx^2\right)\\ &= -\left(x^2\cos x - \int 2x\cos x\, dx\right) = 2\int x\cos x\, dx - x^2\cos x.\end{aligned}$$

Note that we have reduced the original problem to a simpler one, namely finding the integral of $x\cos x$ which can be solved by another application of IBP, see Example 6.1.2 and so,

$$\begin{aligned}\int x^2\sin x\, dx &= -x^2\cos x + 2\left(x\sin x + \cos x\right) + C\\ &= 2x\sin x + 2\cos x - x^2\cos x + C.\end{aligned}$$

Example 6.1.4. This a tricker example. We compute $\int e^x\cos x\, dx$. Let $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x\, dx$ and we take $v = \sin x$. So IBP yields,

$$\int e^x\cos x\, dx = e^x\sin x - \int e^x\sin x\, dx.$$

At the first sight we are getting nowhere since finding $\int e^x\sin x\, dx$ is as hard as the original problem. However, if we apply IBP to the last integral,

$$\int e^x\sin x\, dx = -e^x\cos x + \int e^x\cos\ dx$$

the original integral, call it J, reappears. Thus

$$\begin{aligned}J &= e^x\sin x - (-e^x\cos x + J)\\ 2J &= e^x(\sin x + \cos x) + C\\ J &= \frac{e^x(\sin x + \cos x)}{2} + C.\end{aligned}$$

In some situations IBP yields the so-called **reduction formulas**. For instance,

$$\begin{aligned}
\int \sin^n x\,dx &= -\int \sin^{n-1} x\,d\cos x \\
&= -\left(\sin^{n-1} x\cos x - (n-1)\int \sin^{n-2} x\cos^2 x\,dx\right) \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx \\
n\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx.
\end{aligned}$$

Thus

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx. \tag{6.5}$$

Equation (6.5) is a reduction formula because the integrals appear on both sides are of the same type but the one on the right-hand-side is simpler, in this case the integrand has a lower degree. So by applying (6.5), the problem reduces to the one of finding integral of either $\sin x$ (if n is odd) or 1 (if n is even). Both of them we know how to solve. Likewise, there are reduction formulas for the powers of other trigonometric functions (see Exercises).

Exercises

1. Use integration by parts to evaluate the following indefinite integrals.

   (a) $\int 2y\arctan(5y)\,dy$.

   (b) $\int x^2e^{x/7}\,dx$.

   (c) $\int \ln(x^2+23x+60)\,dx$.

   (d) $\int x\cos^2(4x)\,dx$.

   (e) $\int x^2\sin(3x)\,dx$.

   (f) $\int \sqrt{t}\ln(t)\,dt$.

   (g) $\int -3x^4(\ln(x))^2\,dx$.

   (h) $\int 3x5^x\,dx$.

   (i) $\int x^3e^{x^2}\,dx$.

   (j) $\int 7\cos(\ln(x))\,dx$.

   (k) $\int -3t\sec^2(t)\,dt$

2. Evaluate $\int_0^{1/2} \arccos(x)\, dx$.

3. Derive the following reduction formula

   (a) $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$. Use this formula to evaluate $\int t^3 e^t \, dt$.

   (b) $\int \sin^n(x) = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx$. Deduce that

$$\int_0^{\pi/2} \sin^n(x)\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\, dx.$$

6.2 Trigonometric Integrals

In this section we study how to integrate various special kinds of trigonometric functions.

6.2.1 Applications of the product formulas

We begin by studying how to integrate product involving sine and cosine. The idea is to use a suitable product formula to turn the product into a sum. First, recall that

$$2\sin A\cos B \equiv \sin(A+B) + \sin(A-B). \tag{6.6}$$

Thus

$$2\sin mx\cos nx = \sin(m+n)x + \sin(m-n)x$$

and so for $m \neq n$,

$$\begin{aligned}
2\int \sin mx\cos nx\, dx &= \int \sin(m+n)x\, dx + \int \sin(m-n)x\, dx \\
\int \sin mx\cos nx\, dx &= -\frac{1}{2(m+n)}\cos(m+n)x - \frac{1}{2(m-n)}\cos(m-n)x + C.
\end{aligned}$$

Certainly, when $m = n$, $\sin(m-n)x = \sin 0 = 0$, thus

$$\int \sin mx\cos mx\, dx = -\frac{1}{4m}\cos 2mx + C.$$

Example 6.2.1.

$$\begin{aligned}
\int \sin 3x\cos 4x\, dx &= \frac{1}{2}\left(\int \sin(3+4)x\, dx + \int \sin(3-4)x\, dx\right) \\
&= -\frac{1}{14}\cos(7x) + \frac{1}{2}\cos x + C.
\end{aligned}$$

Likewise, the following product formulas handle integrands of the form $\sin mx \sin nx$ and $\cos mx \cos nx$, respectively.

$$2\sin A \sin B \equiv \cos(A-B) - \cos(A+B) \tag{6.7}$$

$$2\cos A \cos B \equiv \cos(A-B) + \cos(A+B) \tag{6.8}$$

Example 6.2.2. By (6.7)

$$\begin{aligned}\int \sin(3x)\sin(4x)\,dx &= \frac{1}{2}\left(\cos(3x-4x)\,dx - \int \cos(3x+4x)\,dx\right)\\ &= \frac{1}{2}\sin(x) - \frac{1}{14}\sin(7x) + C.\end{aligned}$$

Example 6.2.3. By (6.8)

$$\begin{aligned}\int \cos(3x)\cos(4x)\,dx &= \frac{1}{2}\left(\int \cos(3x-4x)\,dx + \int \cos(3x+4x)\,dx\right)\\ &= \frac{1}{2}\sin(x) + \frac{1}{14}\sin(7x) + C.\end{aligned}$$

Sometime, we need to apply the technique more than once.

Example 6.2.4.

$$\begin{aligned}\int \sin^2 x\cos x\,dx = \frac{1}{2}\int \sin 2x \sin x\,dx &= \frac{1}{4}\int \cos(x) - \cos(3x)\,dx\\ &= \frac{1}{4}\sin x - \frac{1}{12}\sin 3x + C.\end{aligned}$$

In general, by iterating applications these products formula, we can integrate any product of sine and cosine of multiples of x. Next, we will discuss the method which provides a more effective way of integrating such products when all multiples of x involved are the same.

6.2.2 Products of powers of trigonometric functions

To integrate function of the form $\sin^n x \cos^m x$, we consider several cases depending on the parity of m and n. Suppose m is odd, say $m = 2k+1$. Then

$$\begin{aligned}\sin^n x\cos^{(2k+1)} x\,dx &= \sin^n x\cos^{2k} x\cos x\,dx\\ &= \sin^n x(1-\sin^2 x)^k\,d\sin x\end{aligned}$$

By letting $u = \sin x$, the form becomes $u^n(1-u^2)^k\,du$ which can be integrated out readily. For example,

Example 6.2.5.

$$\begin{aligned}\int \sin^4 x \cos^6 x\, dx &= \int u^4(1-u^2)^3\, du = \frac{1}{5}u^5 - \frac{3}{7}u^7 + \frac{3}{9}u^9 - \frac{1}{11}u^{11} + C\\ &= \frac{1}{5}\sin^5 x - \frac{3}{7}\sin^7 x + \frac{1}{3}\sin^9 x - \frac{1}{11}\sin^{11} x + C.\end{aligned}$$

Suppose n is odd, say $n = 2k+1$. The idea for the previous case applies here as well.

$$\begin{aligned}\sin^{(2k+1)} x \cos^m x\, dx &= \sin^{2k} x \cos^m x \sin x\, dx\\ &= -(1-\cos^2 x)^k \cos^m x\, d\cos x\end{aligned}$$

By letting $u = \cos x$, the form becomes $-(1-u^2)^k u^m\, du$ which can be integrated out readily. For example,

Example 6.2.6.

$$\begin{aligned}\int \sin^5 x \cos^2 x\, dx &= \int -(1-u^2)^2 u^2\, du\\ &= \int -u^2 + 2u^4 - u^6\, du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C\\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C\end{aligned}$$

We are left with the case when both m and n are even. By using the identity $\sin^2 x + \cos^2 x \equiv 1$, one can express the integrand as a polynomial in $\cos^2 x$ (or $\sin^2 x$) and we can integrate even powers of $\cos x$ (or $\sin x$) by using double angle formulas to reduce the powers involved. We illustrate this method by the next example.

Example 6.2.7. Let us integrate $\sin^4 x$. Recall that

$$1 - 2\sin^2 A \equiv \cos 2A \equiv 2\cos^2 A - 1.$$

So

$$\begin{aligned}\sin^4 x = (\sin^2 x)^2 &= \left(\frac{1-\cos 2x}{2}\right)^2\\ &= \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right).\end{aligned}$$

Now another application of the double angle formula yields,

$$\cos^2 2x \equiv \frac{1}{2}\left(1 + \cos 2(2x)\right) = \frac{1+\cos 4x}{2}.$$

Thus

$$\int \sin^4 x\, dx = \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) \quad = \frac{1}{8}\int 3 - 4\cos 2x + \cos 4x\, dx$$
$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$$

Next we study how to integrate $\tan^n x \sec^m x$. Again depending on the parity of m and n, it further divides into several cases. Suppose $m > 0$ is even, say $m = 2k+2$. Then

$$\begin{aligned} \tan^n x \sec^{2k+2} x\, dx &= \tan^n x(\sec^2 x)^k \sec^2 x\, dx \\ &= \tan^n x(1+\tan^2 x)^k\, d\tan x \end{aligned}$$

So by letting $u = \tan x$, we turn the form into $u^n(1+u^2)^k\, du$ which can be integrated out readily, For example,

Example 6.2.8.

$$\begin{aligned} \int \tan^3 x \sec^4 x\, dx &= \int u^3(1+u^2)\, du = \int u^3 + u^5\, du \\ &= \frac{1}{4}u^4 + \frac{1}{6}u^6 + C = \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C. \end{aligned}$$

The remaining case for even m is when $m = 0$. In other words, we integrate a power of tangent. For $n \geq 2$,

$$\begin{aligned} \int \tan^n x\, dx &= \int \tan^{n-2} x(\sec^2 x - 1)\, dx \\ &= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx \\ &= \int \tan^{n-2}\, d\tan x - \int \tan^{n-2} x\, dx \\ &= \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx. \end{aligned} \tag{6.9}$$

Using this reduction formula, we can reduced the power of $\tan x$ to 1 and when $n = 1$,

$$\begin{aligned} \int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx = -\int \frac{d\cos x}{\cos x} \\ &= -\ln|\cos x| + C = \ln|\sec x| + C. \end{aligned} \tag{6.10}$$

Example 6.2.9. For example, using the reduction formula (6.9) and (6.10), we get

$$\begin{aligned} \int \tan^3 x\, dx &= \frac{1}{2}\tan^2 x - \int \tan x\, dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C = \frac{1}{2}\tan^2 x + \ln|\cos x| + C. \end{aligned}$$

Next we consider when n is odd. Since we have dealt with the case when $m = 0$, we assume $m \geq 1$, and so

$$\begin{aligned}\tan^{(2k+1)} x \sec^m x \, dx &= \tan^{2k} x \sec^{m-1} x \tan x \sec x \, dx \\ &= (\sec^2 x - 1)^k \sec^{m-1} x \, d \sec x.\end{aligned}$$

By substituting $u = \sec x$, we turn the form into $(u^2 - 1)^k u^{m-1} \, du$ which can be integrated readily. For example,

Example 6.2.10.

$$\begin{aligned}\int \tan^3 x \sec^4 x \, dx = \int (u^2 - 1) u^3 \, du &= \int u^5 - u^3 \, du \\ = \frac{1}{6} u^6 - \frac{1}{4} u^4 + C &= \frac{1}{6} \sec^6 x - \frac{1}{4} \sec^4 x + C.\end{aligned}$$

Comparing with Example 6.2.8, we see that different substitutions can yield different representations of the same class of functions, in this case the class of antiderivatives of $\tan^3 x \sec^4 x$. We encourage the reader to work directly that the two answers are indeed the same class of functions (hint: use the identity $\sec^2 x \equiv \tan^2 + 1$).

The remaining case is when n is even and m is odd. Since an even power of $\tan x$ can be turned into a polynomial in $\sec^2 x$, it suffices to show how to integrate a power (in fact an odd power) of $\sec x$. First,

$$\begin{aligned}\int \sec^m x \, dx &= \int \sec^{m-2} x \sec^2 x \, dx = \int \sec^{m-2} x \, d \tan x \\ &= \sec^{m-2} x \tan x - \int \tan x \, d \sec^{m-2} x \\ &= \sec^{m-2} x \tan x - (m-2) \int \tan x \sec^{m-3} x \sec x \tan x \, dx \\ &= \sec^{m-2} x \tan x - (m-2) \int \tan^2 x \sec^{m-2} x \, dx \\ &= \sec^{m-2} x \tan x - (m-2) \int (\sec^2 x - 1) \sec^{m-2} x \, dx \\ &= \sec^{m-2} x \tan x - (m-2) \int \sec^m x \, dx + (m-2) \int \sec^{m-2} x \, dx.\end{aligned}$$

Hence we obtained the reduction formula, for $m \geq 2$,

$$\int \sec^m x \, dx = \frac{1}{m-1} \sec^{m-2} x \tan x + \frac{m-2}{m-1} \int \sec^{m-2} x \, dx \tag{6.11}$$

which can be used to reduce the power of $\sec x$ down to 1. And we can integrate $\sec x$ as follows:

$$\begin{aligned}\int \sec x\, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C.\end{aligned} \tag{6.12}$$

Example 6.2.11.

$$\begin{aligned}\int \tan^2 x \sec x\, dx &= \int \sec^3 x\, dx - \int \sec x\, dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx - \int \sec x\, dx \\ &= \frac{1}{2}\left(\sec x \tan x - \int \sec x\, dx\right) \\ &= \frac{1}{2}\left(\sec x \tan x - \ln|\sec x + \tan x|\right) + C.\end{aligned}$$

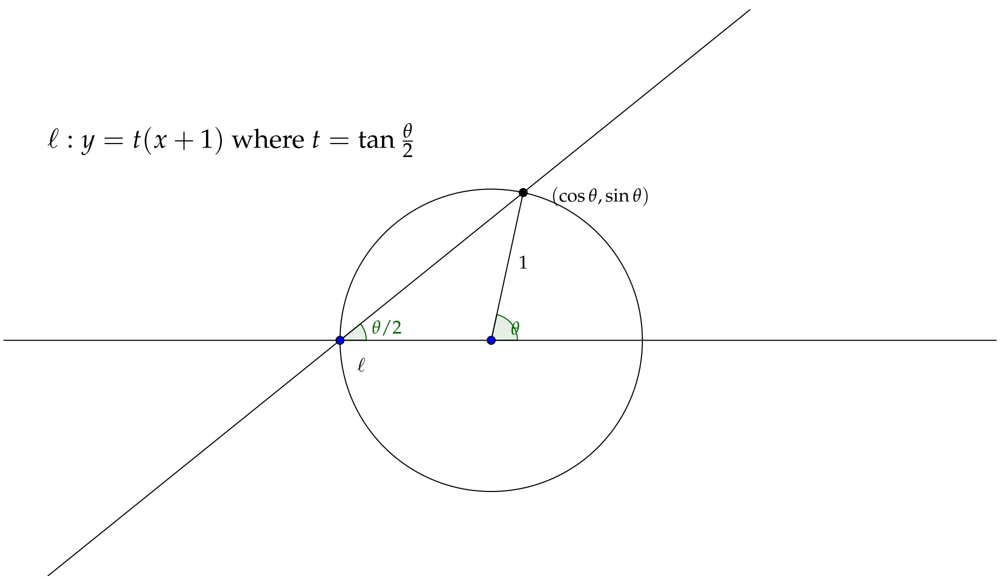
Here is a quick review

Integrand	Substitution
$\sin^{(2k+1)} x \cos^n x\, dx$	$u = \cos x$
$\sin^n x \cos^{(2k+1)} x\, dx$	$u = \sin x$
$\tan^n x \sec^{2k} x\, dx$	$u = \tan x$
$\tan^{(2k+1)} x \sec^n x\, dx$	$u = \sec x$

Table 6.1: Substitutions for integrating various trigonometric functions

6.2.3 Half-angle substitution

There is a way to convert integrals of rational functions in sine and cosine into integrals of rational functions of a single variable. It comes from the following parametrization of the unit circle.



From the picture, one sees that

$$\cos\theta = \frac{1-t^2}{1+t^2}, \qquad \sin\theta = \frac{2t}{1+t^2} \tag{6.13}$$

where $t = \tan(\theta/2)$ is the slope of the line hence the name of **half-angle substitution**. Since

$$dt = d\tan\frac{\theta}{2} = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1}{2}(1+t^2)\,d\theta,$$

so if R is a rational functions in two variables, then

$$R(\cos\theta, \sin\theta)\,d\theta = R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{1+t^2}\,dt.$$

The form on the right-hand side is a rational form in the variable t and can be integrated out by the partial fractions algorithm which we will study in Section 6.4.3. For now, let us just illustrate the ideas by redoing the integral of $\tan\theta$ and $\sec\theta$.

Example 6.2.12. Using the substitution, $t = \tan(\theta/2)$, we have

$$\tan(\theta) = \frac{\sin\theta}{\cos\theta} = \frac{2t}{1-t^2}.$$

Hence

$$\begin{aligned}\int \tan\theta\, d\theta &= \int \frac{2t}{1-t^2}\frac{2}{1+t^2}\, dt \\ &= \int \frac{2}{(1-t^2)(1+t^2)}\, dt^2.\end{aligned}$$

Substitute t^2 by u, we get

$$\begin{aligned}\int \frac{2}{(1-u)(1+u)}\, du &= \int \frac{du}{1+u} + \int \frac{du}{1-u} \\ &= \ln|1+u| - \ln|1-u| + C = \ln\left|\frac{1+t^2}{1-t^2}\right| + C \\ &= \ln|\sec\theta| + C.\end{aligned}$$

Example 6.2.13.

$$\begin{aligned}\int \sec\theta\, d\theta &= \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}\, dt = \int \frac{2}{1-t^2}\, dt \\ &= \int \frac{dt}{1-t} + \int \frac{dt}{1+t} = \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right| + C.\end{aligned}$$

This agrees with the anti-derivative of $\sec\theta$ that we first obtained, since

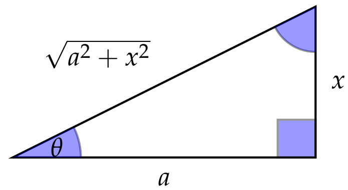
$$\ln\left|\frac{1+t}{1-t}\right| = \ln\left|\frac{(1+t)^2}{1-t^2}\right| = \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| = \ln|\sec\theta + \tan\theta|.$$

6.3 Trigonometric Substitutions

In this section we study integrals involving the square root of quadratic function and their reciprocals. The idea is to convert, via a suitable substitution of the variable by a trigonometric function, the integrand into a trigonometric function which can be integrated by the techniques discussed in the previous section. As we shall see, this idea works for quadratic functions because the Pythagorean theorem imposes quadratic relations on the sides of a right triangle.

By factoring out the leading coefficient and completing the square, we can assume the quadratic function has leading coefficient is 1 and has no linear term. Since we may be taking square root, over real numbers, there are three cases to consider, namely $x^2 + a^2$, $x^2 - a^2$ and $a^2 - x^2$, ($a \in \mathbb{R}$). To each case, we associate a right-triangle which help us to recognize the substitution. Let us start with $x^2 + a^2$. The Pythagorean theorem suggests that we should let x

and a be the two legs of the right triangle,



and the substitution in this case, is $\tan\theta = x/a$ or equivalently, $x = a\tan\theta$, so

$$x^2 + a^2 = a^2(\tan^2\theta + 1) = a^2 \sec\theta,$$

and $dx = \sec^2\theta\, d\theta$.

Example 6.3.1. Let us compute $\int \frac{dx}{(x^2+36)^2}$. The discussion above suggests we should let $x = 6\tan\theta$. So $dx = 6\sec^2\theta\, dt$ and $x^2 + 36 = 36\sec^2\theta$. Therefore,

$$\int \frac{dx}{(x^2+36)^2} = \int \frac{6\sec^2\theta}{(36\sec^2\theta)^2}\, d\theta = \frac{1}{216}\int \cos^2\theta\, d\theta.$$

The last integral can be handle by the technique in Section 6.2

$$\int \cos^2\theta\, d\theta = \int \frac{\cos 2\theta + 1}{2}\, d\theta = \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C$$
$$= \frac{1}{2}(\sin\theta\cos\theta + \theta) + C.$$

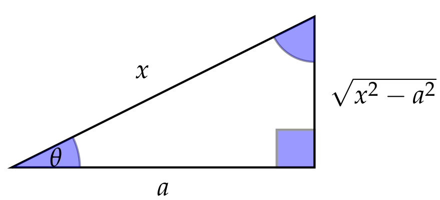
It follows from the diagram that $\theta = \arctan(x/6)$ and that

$$\sin\theta\cos\theta = \frac{x}{\sqrt{x^2+36}}\frac{6}{\sqrt{x^2+36}} = \frac{6x}{x^2+36}.$$

Hence,

$$\int \frac{dx}{(x^2+36)^2} = \frac{1}{432}\left(\frac{6x}{x^2+36} + \arctan\left(\frac{x}{6}\right)\right) + C.$$

Next we examine integrals involving $x^2 - a^2$. The minus sign suggests we should let x be the hypotenuse and a be one of the leg, the associated diagram is



We make the substitution $a/x = \cos\theta$, or equivalently, $x = a\sec\theta$. Then

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$$

and $dx = a\sec\theta\tan\theta\, d\theta$.

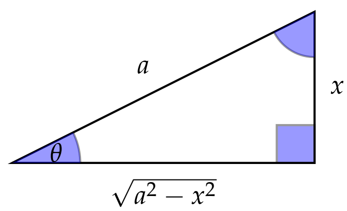
Example 6.3.2. To integrate $\frac{1}{x^2\sqrt{x^2-100}}$, we let $x = 10\sec\theta$, so

$$x^2\sqrt{x^2-100} = (100\sec^2\theta)(10\tan\theta)$$

and $dx = 10\sec\theta\tan\theta\, d\theta$. Thus

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-100}} &= \int \frac{10\sec\theta\tan\theta}{1000\sec^2\theta\tan\theta}\, d\theta = \frac{1}{100}\int \cos\theta\, d\theta\\ &= \frac{\sin\theta}{100} + C = \frac{\sqrt{x^2-100}}{100x} + C.\end{aligned}$$

Finally, let us look at integrals involving $a^2 - x^2$. This time a should be the hypotenuse and x is one of the leg, the associated triangle looks like



Thus the substitution is $x/a = \sin\theta$, or equivalently $x = a\sin\theta$. So

$$a^2 - x^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

and $dx = \cos\theta\, d\theta$.

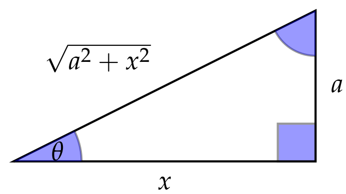
Example 6.3.3. To illustrate this substitution, let us integrate $\frac{1}{\sqrt{49-100x^2}}$. First let $y = 10x$, so

$$\int \frac{dx}{\sqrt{49-100x^2}} = \frac{1}{10}\int \frac{dy}{\sqrt{49-y^2}}.$$

Now let $y = 7\sin\theta$, so $dy = 7\cos\theta\, d\theta$ and the integral becomes

$$\begin{aligned}\frac{1}{10}\int \frac{7\cos\theta}{7\cos\theta}\, d\theta &= \frac{1}{10}t + C = \frac{1}{10}\arcsin\left(\frac{y}{7}\right) + C\\ &= \frac{1}{10}\arcsin\left(\frac{10x}{7}\right) + C.\end{aligned}$$

The reader may note that in each of the associate triangle there are two possible assignments for the legs with respect to the angle θ. For example, we can associate the following triangle



to the case x^2+a^2 instead. This assignment leads to the substitution $x = a\cot\theta$ which also works. Applying this substitution (with $a = 6$) to the integral in Example 6.3.1 yields,

$$\begin{aligned}\int \frac{dx}{(x^2+36)^2} &= \int \frac{d\,6\cot\theta}{(36\cot^2\theta+36)^2} = \int \frac{-6\csc^2\theta}{\left(36(\cot^2\theta+1)\right)^2}\,d\theta \\ &= -\frac{1}{216}\int \frac{\csc^2\theta}{\csc^4\theta}\,d\theta = -\frac{1}{216}\int \sin^2\theta\,d\theta \\ &= -\frac{1}{432}\int (1-\cos 2\theta)\,d\theta = \frac{1}{423}\left(\sin\theta\cos\theta-\theta\right)+C\end{aligned}$$

Express back in terms of x (cf. (6.3.1)), we have

$$\int \frac{dx}{(x^2+36)^2} = \frac{1}{432}\left(\frac{6x}{x^2+36}+\operatorname{arccot}\left(\frac{x}{6}\right)\right)+C.$$

This is the same class of functions given in Example 6.3.1, since for any $y \in \mathbb{R}$, $\arctan y$ and $-\operatorname{arccot} y$ differ by a constant, namely $\arctan y + \operatorname{arccot} y = \pi/2$.

Let us end this section by an example which calls for almost all the techniques that we have discussed so far.

Example 6.3.4. Evaluate the integral $\displaystyle\int \frac{x\arctan(x)}{(1+x^2)^2}\,dx$.

Let $w = 1+x^2$, so $dw = 2xdx$ and the integral becomes $\displaystyle\frac{1}{2}\int \frac{\arctan x}{w^2}\,dw$. Apply IBP to $u = \arctan(x)$ and $dv = dw/w^2$, then $du = dx/(1+x^2)$ and $v = -1/w$. So the integral equals

$$\begin{aligned}&\frac{1}{2}\left(-\frac{\arctan(x)}{w}+\int \frac{1}{w}\frac{dx}{(1+x^2)}\right) \\ =&\frac{1}{2}\left(-\frac{\arctan(x)}{1+x^2}+\int \frac{dx}{(1+x^2)^2}\right).\end{aligned}$$

Now with the trig-substitution $x = \tan\theta$, the last integral

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^4\theta \sec^2\theta \, d\theta = \int \cos^2\theta \, d\theta.$$

Now by the double-angled formula $\cos(2\theta) \equiv 2\cos^2\theta - 1$, the last integral integrated out to

$$\int \frac{1+\cos 2\theta}{2} \, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C = \frac{\arctan(x)}{2} + \frac{x}{2(1+x^2)} + C.$$

Now putting all these back together, we get

$$\begin{aligned}&\frac{1}{2}\left(-\frac{\arctan(x)}{1+x^2} + \frac{\arctan(x)}{2} + \frac{x}{2(1+x^2)}\right) + C\\ =&\frac{\arctan(x)}{4} + \frac{x}{4(1+x^2)} - \frac{\arctan(x)}{2(1+x^2)} + C.\end{aligned}$$

6.4 Rational Functions

We now study how to integrate rational functions. Before getting into the details, a few simplifications are in place. We say that a rational function $p(x)/q(x)$ is **proper** if $\deg p(x) < \deg q(x)$. By the division algorithm, any nonzero rational function is a sum of polynomial and a proper rational function.

Example 6.4.1. The rational function $x^3/(2x-1)^2$ is not proper. Divide x^3 by $(2x-1)^2$, we get

$$x^3 = \frac{1}{4}(x+1)(2x-1)^2 + \left(\frac{3}{4}x + \frac{1}{4}\right)$$

therefore,

$$\frac{x^3}{(2x-1)^2} = \frac{1}{4}(x+1) + \frac{\frac{3}{4}x + \frac{1}{4}}{(2x+1)^2}.$$

Since we already know how to integrate polynomials, let us focus on the proper rational functions. Moreover, we can assume our rational function $p(x)/q(x)$ is **reduced**, i.e. $p(x)$ and $q(x)$ have no non-constant common factors. This is because the function obtained by canceling the non-constant common factors of $p(x)$ and $q(x)$ extends $p(x)/q(x)$. So the integral of $p(x)/q(x)$ is the restriction to its domain of the integral of the reduced function.

Example 6.4.2. The rational function $f(x) = x + 2/(x^2 - 4)$ is not reduced since $x + 2$ and $x^2 - 4$ have a non-constant common factor $x + 2$. By canceling this factor, we get a reduced rational function $r(x) = 1/(x - 2)$ and its indefinite integral is $\ln|x - 2| + C$. We conclude that

$$\int \frac{x+2}{x^2-4}\,dx = \ln|x-2| + C, \qquad (x \neq \pm 2).$$

Next we study integration of rational functions with some restrictions put on their denominators. After that we show how to express a general proper reduced rational function as a sum of rational functions of these special forms.

6.4.1 Special Case: $c/(ax + b)^k$

We begin with two easy examples.

Example 6.4.3. Let us integrate $3/(2x - 1)$. An appropriate substitution in this case would be $u = 2x - 1$. We have $du = 2dx$ and

$$\begin{aligned}\int \frac{3}{2x-1}\,dx &= 3\int \frac{dx}{u} = \frac{3}{2}\int \frac{du}{u} \\ &= \frac{3}{2}\ln|u| + C = \frac{3}{2}\ln|2x-1| + C.\end{aligned}$$

Example 6.4.4. This time we integrate $3/(2x - 1)^4$. Again, substituting $2x - 1$ by u yields

$$\begin{aligned}\int \frac{3}{(2x-1)^4}\,dx &= 3\int \frac{dx}{u^4} = \frac{3}{2}\int \frac{du}{u^4} \\ &= \frac{3}{2}\left(-\frac{1}{3u^3}\right) + C = -\frac{1}{2(2x-1)^3} + C.\end{aligned}$$

From these examples, it is not hard to see how to make use of the substitution $u = ax + b$ to integrate functions of the from $c/(ax + b)^k$ (Exercise).

6.4.2 Special case: $l(x)/q(x)^k$

We now use the techniques developed in Section 6.3 to integrate rational functions of the form $l(x)/q(x)^k$ where $l(x)$ is a linear polynomial and $q(x)$ is an irreducible quadratic polynomial. Clearly, a quadratic polynomial is reducible if and only if it is a product of two linear factors. In other words, a quadratic polynomial $ax^2 + bx + c$ is irreducible over $\mathbb{R}$ if and only if it has no real roots, or equivalently, its discriminant $b^2 - 4ac$ is negative.

Example 6.4.5. Let us compute $\int \frac{dx}{3x^2 - 2x + 4}$. By completing the square,

$$\int \frac{dx}{3x^2 - 2x + 4} = \frac{1}{3} \int \frac{dx}{\left(x - \frac{1}{3}\right)^2 + \frac{11}{9}}.$$

The substitution $u = x - 1/3$ now brings the integral to $\frac{1}{3} \int \frac{du}{u^2 + 11/9}$. An appropriate substitution in this step would be $u = (\sqrt{11}/3) \tan\theta$. This brings us to

$$\begin{aligned} \frac{9}{11} \frac{\sqrt{11}}{3} \frac{1}{3} \int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta &= \frac{\sqrt{11}}{11}\theta + C = \frac{\sqrt{11}}{11} \arctan \frac{3u}{\sqrt{11}} + C \\ &= \frac{\sqrt{11}}{11} \arctan \frac{\sqrt{11}(3x - 1)}{11} + C. \end{aligned}$$

Example 6.4.6. This time we compute $\int \frac{x + 1}{3x^2 - 2x + 4}\, dx$. First note that the derivative of $3x^2 - 2x + 4$ is $6x - 2$, we decompose the integrand accordingly,

$$\begin{aligned} \int \frac{x + 1}{3x^2 - 2x + 4}\, dx &= \frac{1}{6} \int \frac{6x + 6}{3x^2 - 2x + 4}\, dx \\ &= \frac{1}{6} \left(\int \frac{6x - 2}{3x^2 - 2x + 4}\, dx + \int \frac{8}{3x^2 - 2x + 4}\, dx \right) \end{aligned}$$

The first integral can be handled by the substitution $u = 3x^2 - 2x + 4$, and the second integral can be handled the same way as in Example 6.4.5. Putting these together we get,

$$\begin{aligned} \int \frac{dx}{3x^2 - 2x + 4}\, dx &= \frac{1}{6} \left(\int \frac{du}{u} + 8 \int \frac{dx}{3x^2 - 2x + 4} \right) \\ &= \frac{1}{6} \ln|u| + \frac{8}{6} \frac{\sqrt{11}}{11} \arctan \frac{\sqrt{11}(3x - 1)}{11} + C \\ &= \frac{1}{6} \ln(3x^2 - 2x + 4) + \frac{4\sqrt{11}}{33} \arctan \frac{\sqrt{11}(3x - 1)}{11} + C. \end{aligned}$$

We can remove the absolute value because $3x^2 - 2x + 4$ for all $x \in \mathbb{R}$.

It is not really harder when the denominator is a higher power of an irreducible quadratic polynomial.

Example 6.4.7. Compute $\int \frac{dx}{(3x^2 - 2x + 4)^2}$. By completing the square and letting $u = x - 1/3$, we get,

$$\int \frac{dx}{(3x^2 - 2x + 4)^2} = \int \frac{dx}{9\left(\left(x - \frac{1}{3}\right)^2 + \frac{11}{9}\right)^2} = \frac{1}{9} \int \frac{du}{(u^2 + 11/9)^2}.$$

The substitution $u = \left(\sqrt{11}/3\right)\tan\theta$ brings the integral to

$$\frac{1}{9}\frac{\sqrt{11}}{3}\left(\frac{9}{11}\right)^2\int\cos^2\theta\,d\theta = \frac{3\sqrt{11}}{121}\int\cos^2\theta\,d\theta.$$

and

$$\begin{aligned}\int\cos^2\theta\,d\theta &= \int\frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2}+\frac{1}{4}\sin 2\theta + C\\ &= \frac{\theta}{2}+\frac{1}{2}\sin\theta\cos\theta + C.\end{aligned}$$

Expressing the answer back in x,

$$\begin{aligned}&\frac{1}{2}\arctan\frac{3\sqrt{11}u}{11}+\frac{1}{2}\frac{u}{\sqrt{u^2+11/9}}\frac{\sqrt{11}/3}{\sqrt{u^2+11/9}}+C\\ &=\frac{1}{2}\arctan\frac{\sqrt{11}(3x-1)}{11}+\frac{\sqrt{11}}{6}\frac{3x-1}{3x^2-2x+4}+C\end{aligned}$$

Putting these together, we get

$$\int\frac{dx}{(3x^2-2x+4)^2} = \frac{3\sqrt{11}}{242}\arctan\frac{\sqrt{11}\,(3x-1)}{11}+\frac{1}{22}\left(\frac{(3x-1)}{3x^2-2x+4}\right)+C.$$

6.4.3 Partial Fractions Decomposition

A **partial fractions decomposition** (over $\mathbb{R}$) of a proper reduced rational function is an expression of the function as a sum of rational functions of special forms discussed in 6.4.1 and 6.4.2. Such a decomposition is essentially unique because of the unique factorization of polynomials over the real numbers.

Example 6.4.8. To find the partial fractions decomposition of the rational function $f(x) = \dfrac{x-1}{2x^2-x-3}$. First, we need find the irreducible factors of denominator $q(x) = 2x^2-x-3$. Since the discriminant of $q(x)$ is positive, we know that $q(x)$ is reducible. In fact, $q(x) = (x+1)(2x-3)$. We are seeking an expression of $f(x)$ of the form

$$\frac{x-1}{2x^2-x-3} \equiv \frac{A_1}{x+1}+\frac{A_2}{2x-3}$$

where each A_i is a polynomial of degree less than that of the corresponding denominator. Since both denominators are linear, so A_1, A_2 are constant polynomials. The next step is to determine these constants so that the identity holds.

One way to do so is to recombine the sum on the right then compare coefficients. In our case,

$$\frac{A_1}{x+1} + \frac{A_2}{2x-3} \equiv \frac{A_1(2x-3) + A_2(x+1)}{(x+1)(2x-3)}.$$

Comparing the numerator, we have

$$x - 1 \equiv A_1(2x-3) + A_2(x+1) \equiv (2A_1 + A_2)x + (-3A_1 + A_2). \tag{6.14}$$

and so

$$2A_1 + A_2 = 1 \quad \text{and} \quad -3A_1 + A_2 = -1.$$

Solving these linear equations simultaneously yields $A_1 = 2/5, A_2 = 1/5$, therefore the partial fractions decomposition of $f(x)$ is

$$\frac{2/5}{x+1} + \frac{1/5}{2x-3} = \frac{2}{5(x+1)} + \frac{1}{5(2x-3)}.$$

Sometime by setting x to appropriate values may provide a faster way to determine the A_i's. For example, setting $x = -1$ in 6.14, we get $-1 - 1 = A_1(2(-1) - 3) + A_2(-1+1)$, so $-2 = -5A_1$, i.e. $A_1 = 2/5$. Likewise, by setting $x = 3/2$, we get $1/2 = (5/2)A_2$ so $A_2 = 1/5$.

The partial decomposition of $f(x)$ allows us to find its integral using the technique developed earlier.

$$\begin{aligned}\int \frac{x-1}{2x^2 - x - 3}\, dx &= \int \frac{2}{5(x+1)}\, dx + \int \frac{1}{5(2x-3)}\, dx \\ &= \frac{2}{5} \ln|x+1| + \frac{1}{10} \ln|2x-3| + C.\end{aligned}$$

Example 6.4.9. Find the partial fractions decomposition of the rational function $f(x) = \dfrac{x-1}{2x^3 + x^2 - 4x - 3}$. The denominator $q(x) = 2x^3 + x^2 - 4x - 3$ has degree 3 so we know that it is reducible over $\mathbb{R}$. As we have pointed out, finding its factors may not be always easy. Luckily, in this case, $q(x)$ has rational roots -1 and $3/2$ which can be located by the rational root test. Dividing $q(x)$ by the factor $x+1$, we get $2x^2 - x - 3$ which can be further factorized as $(x+1)(2x-3)$. Thus

$$q(x) = (x+1)^2(2x-3)$$

Note that the irreducible factor $x+1$ has multiplicity 2 and so the partial fractions decomposition of $f(x)$ takes the form

$$\frac{x-1}{2x^3 + x^2 - 4x - 3} \equiv \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{2x-3}. \tag{6.15}$$

Comparing the numerator yields,

$$x - 1 \equiv A_1(x+1)(2x-3) + A_2(2x-3) + A_3(x+1)^2$$

Setting $x = -1$ and $x = 3/2$, we get

$$-2 = -5A_2 \quad \text{and} \quad \frac{1}{2} = A_3\left(\frac{5}{2}\right)^2,$$

respectively. Thus $A_2 = 2/5$ and $A_3 = 2/25$. To determine A_1, we get set x to a value other than the previous two, for example, letting $x = 1$ yields

$$0 = -2A_1 - A_2 + 4A_3 = -2A_1 - \frac{2}{5} + \frac{8}{25}$$

and so $A_1 = -1/25$. Therefore, the partial fraction decomposition of $f(x)$ is

$$-\frac{1}{25(x+1)} + \frac{2}{5(x+1)^2} + \frac{2}{25(2x-3)}$$

In general the form of the partial fractions decomposition of $p(x)/q(x)$ is determined as follows: each irreducible factor $s(x)$ of $q(x)$ contributes the following sum

$$\frac{r_1(x)}{s(x)} + \frac{r_2(x)}{s(x)^2} + \cdots + \frac{r_d(x)}{s(x)^d}$$

to the decomposition where d is the multiplicity of $s(x)$ in the factorization of $q(x)$ into product of irreducible polynomials. If $s(x)$ is linear then each $r_i(x)$ is a constant to be determined or else $s(x)$ is quadratic then each $r_i(x)$ takes the form a linear polynomial of which the coefficients remain to be determined.

Example 6.4.10. The partial fraction decomposition of $f(x) = \dfrac{x^3 + x^2 + x - 3}{(x-2)^3(x^2+1)^2}$ has the form

$$\frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{B_1 + C_1x}{x^2+1} + \frac{B_2 + C_2x}{(x^2+1)^2}. \tag{6.16}$$

Let us see how to determine these coefficients. If we compare the numerators of both sides, we get

$$\begin{aligned} x^3 + x^2 + x - 3 &\equiv A_1(x-2)^2(x^2+1)^2 + A_2(x-2)(x^2+1)^2 + A_3(x^2+1)^2 \\ &\quad + (B_1 + C_1x)(x^2+1)(x-2)^3 + (B_2 + C_2x)(x-2)^3 \\ &\equiv (x^2+1)^2(A_1(x-2)^2 + A_2(x-2) + A_3) + (x-2)^3v(x) \\ &\equiv (x^2+1)^2u(x) + (x-2)^3v(x) \end{aligned} \tag{6.17}$$

where $u(x) = A_1(x-2)^2 + A_2(x-2) + A_3$ and $v(x) = (B_1 + C_1 x)(x^2+1) + (B_2 + C_2 x)$. To determine the A_i's, first we set $x = 2$ in (6.17) and get $11 = 25A_3$, thus $A_3 = 11/25$. Next we differentiate both sides (6.17) respect to x,

$$3x^2 + 2x + 1 \equiv (x^2+1)^2 u'(x) + 4x(x^2+1)u(x) + (x-2)^2(\cdots) \tag{6.18}$$

then by setting $x = 2$, we get $17 = 25u'(2) + 40u(2)$. Since $u'(x) = 2(x-2)A_1 + A_2$, so $u'(2) = A_2$ and $u(2) = A_3$, therefore, $17 = 25A_2 + 40(11/25)$, so $A_2 = -3/125$. To find A_1, we differentiate (6.18) with respect to x again,

$$6x + 2 \equiv (x^2+1)^2 u''(x) + 8x(x^2+1)u'(x) + 4(3x^2+1)u(x) + (x-2)(\cdots) \tag{6.19}$$

As before by setting $x = 2$ and solve for A_1, we find that $A_1 = -87/625$. Now we are going to determine the rest of the constants. Comparing the coefficient of the highest term, namely x^6, on both sides of (6.17), we get $A_1 + C_1 = 0$, hence $C_1 = 87/625$. Setting $x = 0$ in (6.16), we get

$$-\frac{A_1}{2} + \frac{A_2}{4} - \frac{A_3}{8} + B_1 + B_2 = \frac{3}{8} \tag{6.20}$$

By substituting the known values of the A_i's, we get $B_1 + B_2 = 229/625$. Next by setting $x = \pm 1$ in (6.16), we get

$$\begin{aligned} -A_1 + A_2 - A_3 + \frac{B_1 + C_1}{2} + \frac{B_2 + C_2}{4} &= 0 \\ \frac{B_1 + C_1}{2} + \frac{B_2 + C_2}{4} &= \frac{203}{625}. \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} -\frac{A_1}{3} + \frac{A_2}{9} - \frac{A_3}{27} + \frac{B_1 - C_1}{2} + \frac{B_2 - C_2}{4} &= \frac{1}{27} \\ \frac{B_1 - C_1}{2} + \frac{B_2 - C_2}{4} &= \frac{6}{625}. \end{aligned} \tag{6.22}$$

Solving the system consists of equations (6.21), (6.22), $B_1 + B_2 = 229/625$ and $C_1 = 87/625$ yields $B_1 = 189/625, B_2 = 8/125$ and $C_2 = 44/125$. Therefore, the partial fractions decomposition of $f(x)$ is

$$-\frac{87}{625(x-2)} + \frac{3}{125(x-2)^2} + \frac{11}{25(x-2)^3} + \frac{189 + 87x}{625(x^2+1)} + \frac{8 + 44x}{125(x^2+1)^2}.$$

Exercises

1. Verify that for $a \neq 0$,

$$\int \frac{c}{(ax+b)^k}\,dx = \begin{cases} \dfrac{c}{a}\ln|ax+b| + C & k = 1, \\ \\ -\dfrac{c}{a(k-1)(ax+b)^{k-1}} + C & k > 1. \end{cases}$$

Chapter 7

Improper Integrals

7.1 Definitions

The theory of definite integrals that we have developed so far only deals with bounded functions on bounded intervals. Does it make sense to talk about definite integrals of functions which fail one or both of these requirements? For exmaple, does it make sense to consider the integral of the function e^{-x} on the interval $[1, \infty)$. A moment of thought should convince the reader that the area between the graph of this function and the x-axis should be finite since it is less than the area represented by the geometric series

$$\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots$$

which has a finite sum, namely $1/(e-1)$.

insert a diagram here.

Also, by reflecting the graph of e^{-x} along the line $y = x$, we obtain the graph of its inverse, namely $\ln(1/x)$, which is unbounded on the interval $(0, 1/e]$. Since reflections preserve area, the integrals $\int_1^\infty e^{-x}\,dx$ and $\int_0^{1/e} \ln(1/x)\,dx$ should be the same. Now what should be their common value? Note that for any fix $t > 1$, $\int_1^t e^{-x}\,dx = 1/e - 1/e^t$ and as the limit

$$\lim_{t\to+\infty} \left(\frac{1}{e} - \frac{1}{e^t}\right) = \frac{1}{e}$$

exists, it makes sense to assign $1/e$ as the value of $\int_1^\infty e^{-x}\,dx$ and doing so will be consistent with the interpretation of definite integrals of a positive function as the “area” underneath the graph.

In general, suppose f is a function defined on $[a, b)$ where b is finite of $+\infty$. Moreover, suppose

1. f is integrable on $[a, t]$ for each $a < t < b$; and that
2. the limit $\lim_{t \to b^-} \int_a^t f(x)\, dx$ exists in $\mathbb{R} \cup \{\pm\infty\}$. Then we define

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx. \tag{7.1}$$

Note that, by the fundamental theorem of calculus, this definition agrees with the original definition definite integral when $b \in \mathbb{R}$ and f is integrable on $[a, b]$. If $b = \infty$ or f is not integrable on $[a, b]$, for example f is unbounded on $[a, b]$, then we call the integral in (7.1) an **improper integral**.

Likewise, if f is defined on $(a, b]$ where $a \in \mathbb{R} \cup \{-\infty\}$ and f is integrable on $[c, b]$ for all $a < t < b$ and then we define

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx \tag{7.2}$$

whenever the limit exists. Finally if f is defined on (a, b) where $a < b$ and $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and integrable on all subintervals $[s, t]$, then we fix $c \in (a, b)$ and define

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \tag{7.3}$$

provided that the integrals on the right-hand side exist, possibly as improper integrals, and that the sum is not of the form $+\infty + (-\infty)$. We leave it as an exercise for the reader to justify that the definition is independent of the choice of c. We say that an improper integral **converges** if it is has a finite value, if it is either $+\infty$ or $-\infty$, then we say that the improper integral **exists** but **diverges** to $+\infty$ or $-\infty$, respectively. Now let us work out a few examples to solidify these concepts.

Example 7.1.1. For $t > 1$, the definite integral

$$\int_1^t \frac{1}{x}\, dx = \ln t - \ln 1 = \ln t \to +\infty,$$

as $t \to +\infty$. Therefore, the improper integral $\int_1^\infty \frac{1}{x}\, dx$ exists but diverges to $+\infty$. For $0 < t < 1$, the integral

$$\int_t^1 \frac{1}{x}\, dx = \ln 1 - \ln t = \ln \frac{1}{t} \to +\infty$$

as $t \to 0^-$. Thus the improper integral $\int_0^1 \frac{1}{x}\, dx$ also diverges to infinity. Thus the improper integral $\int_0^\infty \frac{1}{x}\, dx$ does not exist.

Example 7.1.2. For $t > 1$, the definite integral

$$\int_1^t \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_1^t = 1 - \frac{1}{t} \to 1$$

as $t \to \infty$. Thus the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}$ converges to 1. In fact, the power 2 is not that special, in order for the improper integral to converge all we need is the power $p > 1$, since

$$\int_1^t \frac{1}{x^p}\, dx = \frac{1}{(1-p)x^{p-1}}\bigg|_1^t = \frac{1}{p-1} - \frac{1}{(p-1)t^{p-1}} \to \frac{1}{p-1}$$

as $t \to \infty$. On the other hand, it is easy to check that $\int_1^\infty x^{-p}\, dx = +\infty$ when $0 < p < 1$.

Example 7.1.3. For $t > 0$, the integral $\int_0^t \sin x\, dx = 1 - \cos t$ which oscillates between 0 and 2, as $t \to \infty$, therefore the limit $\lim_{t\to\infty} \int_0^t \sin x\, dx$ does not exist and hence $\int_0^\infty \sin x\, dx$ diverges and has no meaning. Similarly, $\int_{-\infty}^0 \sin x\, dx$ and $\int_{-\infty}^\infty \sin x\, dx$ have no meaning. However, because $\sin x$ is an odd function, the limit

$$\lim_{t\to\infty} \int_{-t}^t \sin x\, dx$$

clearly exists and equals 0. This shows that the existence of the limit $\displaystyle\lim_{t\to\infty}\int_{-t}^t f(x)\, dx$ does not guarantee the existence of the improper integral $\int_{-\infty}^\infty f(x)\, dx$.

7.2 Comparison Test

Theorem 7.2.1. *Suppose $g \le f \le h$ and integrable on $[c, d]$ then*

$$\int_c^d g \le \int_c^d f \le \int_c^d h.$$

By take limits, we see that Theorem 7.2.1 still holds for improper integrals. In particular, we have

Corollary 7.2.2.

1. *If $0 \le f \le h$ and $\int_a^b h$ converges then so is $\int_a^b f$.*
2. *If $\int_a^b g = +\infty$ then so is $\int_a^b f$.*

Chapter 8

Sequences

The study of infinite sequences and series are fundamental to the theory of Calculus. We will cover the basics about sequences and series in this chapter and the next.

8.1 The Basics

A **sequence** is simply a list of objects (e.g. real numbers). Usually, an infinite list is indexed by $\mathbb{N}$ the set of natural numbers but in fact another other set that has the same (order) type as $\mathbb{N}$ will do as well. For example, we may use

$$I = \{-3, -2, -, 1, 0, 1, 2, \dots\}$$

to index a sequence $(a_i)_{i\in I}$ of real numbers, so a_{-3} is the first term of the sequence. By the **general term** of a sequence we mean a description of the term as a function of the index. For example, $a_n = 2n$ is the general term of the sequence of even numbers. As a function, the general term of a sequence need not be specified by a formula. For instance, f_n, the n-th Fibonacci number can be defined recursively as $f_0 = 1, f_1 = 1, f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.

Example 8.1.1. A **constant sequence** is a sequence in which every term has the same value. The simplest example of such is probably the **zero sequence**.

$$0, 0, 0, \dots$$

A sequence of real numbers (a_i) is **bounded** if there is $M \in \mathbb{R}$ such that $a_i \leq M$ for each $i \in I$. Similarly, we say that (a_i) is **bounded** if there is $m \in \mathbb{R}$ such that $m \leq a_i$ for all $i \in I$. We say that (a_i) is **bounded** if it is both bounded above and below. In other words, the set of terms $\{a_i : i \in I\}$ of the sequence is a contained in an interval $[m, M]$ for some $M, m \in \mathbb{R}$.

Example 8.1.2. Suppose $a_k = (-1)^k$, then $(a_k)_{k \geq 0}$ is the sequence

$$1, -1, 1, -1, 1, -1, \ldots$$

that is clearly bounded. Note that the set of terms $\{a_k : k \geq 0\}$ is simply $\{1, -1\}$.

A real sequence (a_i) is **increasing** if $a_j \leq a_k$ whenever $j, k \in I$ and $j \leq k$. Likewise, (a_i) is **decreasing** if $a_j \geq a_k$ whenever $j \leq k$. A **monotone** sequence is a sequence that is either increasing or decreasing.

Example 8.1.3. The sequence $(1/n)_{n \geq 1}$ is decreasing and bounded below (e.g. by 0). The sequence of (n^2) are both increasing and unbounded. The sequence $(1 - 1/n)$ is increasing and bounded above (e.g. by 1). The sequence $\left(\sin \frac{n\pi}{2}\right)_{n \geq 0}$ is

$$0, 1, 0, -1, 0, 1, 0, -1, \ldots$$

which is neither increasing nor decreasing. The only sequences that are both increasing and decreasing are the constant sequences.

8.2 Convergence

I

In Calculus or Analysis, one usually interests in the eventual behavior of a sequence. This leads to an important concept called the limit of a sequence. First, we say that a sequence $(a_i)_{i \in I}$ **converges to 0** if for any $\varepsilon > 0$, there exists $n_0 \in I$ such that $|a_i| < \varepsilon$ whenever $i > n_0$. Intuitively, a sequence converges to 0 means eventually its terms all cluster around the origin.

Example 8.2.1. The sequence $(1/n)_{n \geq 1}$ converges to 0. To see this, pick any $\varepsilon > 0$, we can then take n_0 to be any integer bigger than $1/\varepsilon$, e.g. take $n_0 = \lfloor 1/\varepsilon \rfloor + 1$. Then for any $n > n_0$, $|1/n| = 1/n < 1/n_0 < \varepsilon$.

More generally, for a real sequence $(a_i)_{i \in I}$, we say that a real number L is a **limit** of $(a_i)_{i \in I}$ if the sequence $(a_i - L)_{i \in I}$ converges to 0. In this case, we say that (a_i) **converges to** L and denoted by

$$\lim_{i \to \infty} a_i = L \quad \text{or} \quad a_i \to L \text{ as } i \to \infty$$

The limit of a convergent real sequence is unique, to see this, suppose L and L' are limits of the sequence (a_i), so for any $\varepsilon > 0$, there exist N and N' such that $|a_i - L| < \varepsilon$ and $|a_i - L'| < \varepsilon$. Thus for $i > \max N, N'$

$$0 \leq |L - L'| \leq |a_i - L| + |a_i - L'| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we must have $L = L'$.

Example 8.2.2. The sequence $(1 - 1/n)$ converges to 1, since

$$\left(1 - \frac{1}{n}\right) - 1 = -\frac{1}{n} \to 0$$

as $n \to \infty$.

A sequence is **convergent** if it converges to some real number L. So the "picture" of a convergent sequence is the same as one that converges to 0, except the origin is translated to L. We say that a sequence is **divergent** if it is not convergent. We say that a sequence (a_i) **diverges to** $+\infty$ if for any real number U, there exists $n_0 \in I$ such that $a_i > U$ for all $i > n_0$. In other words, given any bound B, eventually every term of the sequence will be bigger than B. Similarly, (a_i) **diverges to** $-\infty$ if for any real number L, there exists $n_0 \in I$ such that $a_i < L$ for all $i > n_0$.

Proposition 8.2.3. *Every convergent sequence is bounded.*

Proof. Suppose (a_i) converges to some real number L. Then for some $n_0 \in I$ we have $|a_i - L| < 1$. In other words, $a_i \in [L-1, L+1]$ for all $i > n_0$. □

Example 8.2.4. The sequence $((-1)^n)_{n \geq \mathbb{N}}$ diverges. Intuitively, the terms of this sequence are flip-flopping between 1 and -1 and so never get close to a fix real number. To be more rigorous, suppose L is the limit, for n sufficiently large, we have $|(-1)^k - L| < 1/2$ for all k sufficiently large. So we have both $|1 - L|$ and $|-1 - L| < 1/2$ but this is absurd since the distance between 1 and -1 is bigger than 1.

The operations on real numbers extend naturally to real sequences, moreover these operations respect limits of convergent sequences.

Proposition 8.2.5. *Suppose (a_i) and (b_i) are convergent seqeuences with limit A and B, respectively. Then the seqeuences $(a_i + b_i)$, $(a_i b_i)$ are convergent. Moreover, if $b_i \neq 0$ for all i and $B \neq 0$, then $(1/b_i)$ is convergent as well. The limits of these sequences are indicated below.*

1. $\lim\limits_{i \to \infty} a_i + b_i = A + B.$

2. $\lim\limits_{i\to\infty} a_i b_i = AB$.

3. $\lim_{i\to\infty}(1/b_i) = 1/B$.

If one takes (b_i) to be a constant sequence, i.e. each $b_i = c$ for some constant c, then Proposition 8.2.5 (2) reads $\lim_{i\to\infty} ca_i = c\lim_{i\to\infty} a_i$. Also from (2) and (3), one gets $\lim_{i\to\infty}(a_i/b_i) = A/B$ provided that each $b_i \neq 0$ and $B = \lim_{i\to\infty} b_i \neq 0$.

Example 8.2.6. Consider the sequence with general term

$$\frac{2 - 3\sin n + 5n^2}{3n - 2n^2}$$

Since

$$\frac{2}{3n-2n^2} \to 0, \quad \frac{3\sin n}{3n-2n^2} \to 0, \quad \frac{5n^2}{3n-2n^2} = \frac{5}{3/n-2} \to \frac{5}{-2}$$

as $n \to \infty$. According to Proposition 8.2.5, we conclude that the original sequence has limit $-5/2$.

Care should be taken when one tries to infer limits using Proposition 8.2.5. For instance, if the proposition were true for divergent sequences then by taking $a_i = i + k$ and $b_i = -i$ where k is a fix by arbitrary constant, then we would have $k = \infty + (-\infty)$ which is absurd.

8.3 Tests of Convergence

In many situations, we are just interested in whether a sequence is convergent or not but not the actual limit itself. However, so far we have no way to verify convergency unless we have the right guess of the limit in advance. In this section, we will discuss a few tests of convergence which do not require "knowing" the limit in advance. All of them rely on the so-called completeness of real numbers. First, we begin by identifying a property of convergent sequences: a sequence (a_i) is **Cauchy** if for any $\varepsilon > 0$ there exists $n_0 \in I$ such that for all $j, k > n_0$, $|a_i - a_k| < \varepsilon$.

Proposition 8.3.1. *Every convergent sequence is Cauchy.*

This is quite clear since if all terms of the sequences are closed to the limit eventually, then they are close to each other eventually.

Proof. Suppose (a_i) is a convergent sequence. Let $L \in \mathbb{R}$ be its limit. Then for any $\varepsilon > 0$, there exists $n_0 \in I$ such that $|a_i - L| < \varepsilon/2$ whenever $i > n_0$. Thus for $j, k > n_0$, we have

$$|a_j - a_k| \leq |a_j - L| + |a_k - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

hence (a_i) is a Cauchy sequence. □

The fact that every real Cauchy sequence converges, which is the converse of Proposition 8.3.1, is also known as the **completeness of real numbers**. Depending on ones definition of the real numbers, this statement can either be proved or taken as a one of the defining properties of the reals.

Example 8.3.2. To say that completeness is something special about the real numbers, consider the following sequence of rational numbers.

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

which is the sequence of decimal approximations of $\sqrt{2}$ to more and more accuracy. The sequence is cleary Cauchy because for any $n \leq m$, the distance between the n-th and m-th term of the sequence is less than 10^{n-1}. However, the sequence does not converge in $\mathbb{Q}$ if so then the rational limit, as a real number, would be the limit of the sequence in $\mathbb{R}$ which is $\sqrt{2}$, contradicting the fact that $\sqrt{2}$ is not rational.

Roughly speaking, completeness means there are no "holes" on the real number, in contrast there are a lot of "holes" among the rationals. And in a sense that one can make precise, the real numbers is the completion of the rationals with respect to the normal distance between numbers. There is another intuitive form of the completeness of reals: *A sequence that is both increasing (decreasing) and bouned above (below) must be convergent*.

Example 8.3.3. The sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots, \frac{n}{n+1} + \ldots$$

is clearly both increasing and bounded above, say by 10, hence convergent. In fact, it is not hard to see that 1 is the limit of the sequence.

Example 8.3.4. The sequence $(\ln k/k)_{k \geq 1}$ is decreasing and bounded below hence convergent. It is clear that 0 is a lower bound of the sequence. It is decreasing because the function $\ln x/x$ is decreasing on $(1, \infty)$. To see that simply note that the derivative of the function is $(1 - \ln x)/x^2$ which is negative for $x > 1$.

In fact, the limit of the sequence is 0. One way to see this is to show that $\lim_{x\to\infty} \ln x/x = 0$ by L'Hospital rule. Another way is to note that when $k > e$, $\sqrt{x} > \ln x$ and hence $0 \le \ln(k)/k < 1\sqrt{k}$. Since the latter sequence converges to 0, this forces $\ln(k)/k \to 0$ as $k \to \infty$.

The last arguement above is in fact a special case of the following intuitively clear test of convergence (divergence).

Proposition 8.3.5 (Sandwich Test)**.** *Suppose (a_i), (b_i) and (c_i) are sequences with $a_i \le c_i \le b_i$ for all sufficiently large i. If*

1. *(a_i) diverges to $+\infty$ then so is (c_i).*
2. *(b_i) diverges to $-\infty$ then so is (c_i).*
3. *(a_i) and (b_i) both converges to some real number L, then so it (c_i).*

Example 8.3.6. Consider the sequence $((2^n - 10)^{-1})_{n\ge 1}$. Since

$$0 \le \frac{1}{2^n - 10} \le \frac{1}{n} \qquad \text{for} \quad n \ge 4$$

and $1/n \to 0$ as $n \to \infty$. By Proposition 8.3.5 (3), we conclude that the sequence $((2^n - 10)^{-1})$ converges to 0.

Exercises

1. Give an example of each of the following
   (a) A non-increasing sequence that is not bounded above.
   (b) A non-decreasing sequence that is not bounded below.
   (c) A sequence that is not bounded above but not diverging to $+\infty$.
   (d) A divergent sequence (a_i) such that the sequence $(|a_i|)$ converges.
2. Show that if (b_i) is a convergent sequence with limit $B \ne 0$, then $b_i \ne 0$ for all i sufficiently large.
3. Show that $|a_i| \to 0$ implies $a_i \to 0$.

Chapter 9

Series

9.1 Convergence

A **series** is an expression of the form

$$a_1 + a_2 + a_3 \cdots \tag{9.1}$$

One can think of it as a sum of the sequence (a_n). More generally, we write $\sum_{i \in I} a_i$ for a series whose terms are indexed by elements of I. Since potentially there are infinitely many non-zero terms in a series, we need to explain the meaning of an expression like (9.1). The idea is simple and by now should be familiar to the reader (c.f. improper integrals): we add one term of the series at a time then take the limit. We call the sum of the first n terms of the series its **n-th partial sum**. If (a_i) is a sequence of real numbers, then we say that the series **converges** to a real number L if, $(s_n)_{n \geq 1}$, its sequence of partial sums converges to L. In this case, we say that L is the **sum** of the series and write this as $\sum_{i \in I} a_i = L$. Likewise, the series diverges (resp. to $+\infty$, $-\infty$), if its sequence of partial sums does.

Example 9.1.1. Consider the series

$$1 - 1 + 1 - 1 + \cdots$$

Its first partial sum s_1 is 1 and the 2nd partial sum is $s_2 = 1 + (-1) = 0$ and so on. Thus its sequence of partial sums is the divergent sequence $1, 0, 1, 0, \cdots$ and hence the series $\sum_{k \geq 0} (-1)^k$ is divergent.

Example 9.1.2. The series $1 + 1 + 1 + \cdots$ diverges to $+\infty$ since its sequence of partial sums is the sequence $1, 2, 3, \ldots$

Proposition 9.1.3. *If $\sum_{i\in I} a_i$ converges then (a_i) converges to 0.*

Proof. For any $\varepsilon > 0$, since the sequence of partial sums converges, so there exists $n_0 \in I$ such that for all $i > n_0$, $|s_i - L| < \varepsilon/2$. In particular,

$$|a_j| = |s_j - s_{j-1}| \leq |s_j - L| + |s_{j-1} - L| < \varepsilon$$

for all $j > n_0 + 1$. This establishes the assertion. □

Often Proposition 9.1.3 is used to show divergence.

Example 9.1.4. By Proposition 9.1.3, the series

$$\sum_{n\geq 1} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

diverges since its terms $n/(n+1)$ tends to 1 instead of 0.

Example 9.1.5. The series $\sum_{k\geq 1} \sin k$ diverges since its sequence of terms $(\sin k)$ is divergent.

Example 9.1.6. The series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

has

$$\frac{1}{2}, \frac{3}{4}, \cdots \frac{2^n - 1}{2^n}, \cdots$$

as its sequence of partial sums which converges to 1. So the sum of the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

The series appears in the example above is a geometric series. In general, a **geometric series** is a series of the form

$$a + ar + ar^2 + \cdots + ar^n + \cdots \tag{9.2}$$

where $a, r \in \mathbb{R}$, $a \neq 0, r \neq 0, 1$. We call a the initial term and r the common ratio of the geometric series (9.2). The trick to finding the sum of a geometric series is to note that

$$rs_n = r(a + ar + \cdots + ar^n) = a(r + r^2 + \cdots + r^{n+1})$$

hence the terms, except the first and the last, in $s_n - rs_n$ canceled out. That is $s_n(1-r) = a(1 - r^{n+1})$, therefore s_n, the n-th partial sum of the series is $a(1 - r^{n+1})/(1-r)$. Since r^n converges to 0 if and only if $|r| < 1$, we conclude that

Proposition 9.1.7. *The geometric series* $\sum_{k=0}^{\infty} ar^k$ *converges to* $\frac{a}{1-r}$ *if* $|r| < 1$ *and diverges otherwise.*

Example 9.1.8. Given that the following series is geometric

$$\frac{2}{5} - \frac{3}{10} + \frac{9}{40} - \cdots$$

then the initial term is $2/5$ and since the common ratio $-3/4$ has absolute value less than 1, according to Proposition 9.1.7, the series converges to $(2/5)/(1 - (-3/4)) = 8/35$.

Another type of series that we can find the sum easily are those with "telescopic" partial sums.

Example 9.1.9. Find the sum of

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots$$

The partial fraction decomposition of the general term is

$$\frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$$

Hence, as $n \to \infty$, the n-th partial sum tends

$$\begin{aligned} s_n &= \frac{1}{2}\Big[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \\ &\qquad \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n-2}\right)\Big] \\ &= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] \longrightarrow \frac{1}{2}\left[\frac{3}{2}\right] = \frac{3}{4}. \end{aligned}$$

Therefore, $\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{3}{4}$.

There is a stronger notion of convergence for series. A series $\sum_{i \in I} a_i$ is **absolutely convergent**, if the series $\sum_{i \in I} |a_i|$ is convergent. Using the triangle inequality, it is not hard to show that absolute convergent series is convergent. On the other hand, a convergent series needs not converge absolutely. As we shall see, for example, the series $\sum_{k \geq 0} \frac{(-1)^k}{(k+1)}$ converges (Example 9.2.13) however, $\sum_{k \geq 0} \left|\frac{(-1)^k}{(k+1)}\right| = \sum_{k \geq 0} \frac{1}{k+1}$ is the harmonic series which is divergent (see Example 9.2.2).

9.2 Convergence Tests

By interpreting the sum of a series with positive terms as area, we can compare it with an improper integral.

Theorem 9.2.1 (Integral Test)**.** *Suppose f is a continuous, decreasing function on an interval $[a, \infty)$, then the series $\sum_{k \geq a} f(k)$ converges if and only if the improper integral $\int_a^\infty f(x)\, dx$ converges.*

Example 9.2.2. The **Harmonic series**

$$\sum_{k\geq 1} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is divergent. This follows by applying Theorem 9.2.1 to the function $f(x) = 1/x$ which is continuous and decreasing on $[1, \infty)$ and that the improper integral $\int_1^\infty \frac{1}{x}\, dx$ is divergent (Example 7.1.1).

include a picture.

Example 9.2.3. The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$$

is convergent. Again, this follows from Theorem 9.2.1 since $1/x^2$ is continuous, decreasing on $[1, \infty)$ and that $\int_1^\infty \frac{1}{x^2}\, dx$ is convergent. The sum of this series is $\frac{\pi^2}{6}$. The appearance of π here is rather interesting. One way to show this is by the residue theorem in Complex Analysis (give a reference).

By the same argument in the previous example, one shows that

Proposition 9.2.4. *For $p \geq 0$, the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, converges if and only if $p > 1$.*

Example 9.2.5. Since $1/(k^2+k) < 1/k^2$ for all $k \geq 1$, by Proposition 9.2.4 and Theorem 9.2.6 we conclude that the series $\sum_{k\geq 1} \frac{1}{k^2+k}$ is convergent. Note that the series has telescopic partial sums, we leave it as an exercise to the reader to find its sum.

include a pic

By applying the comparison test for sequences to the sequence of partial sums, we obtain:

Theorem 9.2.6 (Comparison Test for series)**.** *Suppose $a_i \leq c_i \leq b_i$ for all $i \in I$. Then*

1. *$\sum c_i$ diverges to $+\infty$ if $\sum a_i$ does.*
2. *$\sum c_i$ diverges to $-\infty$ if $\sum b_i$ does.*
3. *$\sum c_i = L$ if both $\sum a_i$ and $\sum b_i$ converges to L.*

We can use geometric series as a reference for comparison, we obtain the next two tests of convergence.

Theorem 9.2.7 (Ratio Test)**.** *Suppose the sequence of ratios $\left|\frac{a_{i+1}}{a_i}\right|$ tends to some real number r, then the series $\sum_{i \in I} a_i$*

1. *converges absolutely, if $r < 1$.*
2. *diverges, if $r > 1$.*

The test is inconclusive if $r = 1$.

Here we only prove the convergence statement.

Proof. By our assumption on the ratios, there exists $i_0 \in I$, such that

$$\left|\frac{a_{i+1}}{a_i}\right| < R := \frac{1+r}{2} < 1,$$

whenever $i \geq i_0$. From this it follows that $|a_{i_0+m}| \leq a_{i_0} R^m$. Since $R < 1$, the geometric series $\sum_{m \geq 0} a_{i_0} R^m$ converges and so by the comparison test, the series $\sum_{i \geq i_0} a_i$ and hence the original series converges absolutely. □

Another way of comparing a series with a geometric series lead us to

Theorem 9.2.8 (Root Test)**.** *Suppose $|a_k|^{1/k}$ tends to some real number r then the series $\sum a_i$*

1. *converges absolutely, if $r < 1$.*
2. *diverges, if $r > 1$.*

The test is inconclusive if $r = 1$.

Again we only prove the convergence statement here.

Proof. This time $|a_k|^{1/k} < R := (1+r)/2 < 1$ for all k sufficiently large. We conclude that the series $\sum_{k\geq k_0} a_k$ converges absolutely by comparing it with the geometric series $\sum_{k\geq k_0} R^k$. $\square$

There are more precise statements of the ratio test and the root test involving limit supremum and limit infimum. We outline their proofs in the exercises.

Example 9.2.9. Consider the series $\sum \frac{k}{5^k}$. Let $a_k = k/5^k$, then as $k \to \infty$

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{k+1}{5^{k+1}}\frac{5^k}{k} = \frac{k+1}{5k} \to \frac{1}{5}.$$

Hence the series converges absolutely according to the ratio test. Recall that $\lim k^{1/k} = 1$, so the series converges absolutely according to the root test as well.

Example 9.2.10. Consider the series $\sum a_k$ where $a_k = \frac{k}{k^2+1}$. Then both $|a_{k+1}/a_k|$ and $|a_k|^{1/k}$ tends to ! as $k \to \infty$. So neither the ratio test nor the root test gives any information. However, for $k \geq 1$

$$\frac{k}{k^2+1} \geq \frac{k}{2k^2} = \frac{1}{2k}$$

Then as the harmonic series diverges, we conclude from the comparison test the $\sum a_k$ diverges.

Example 9.2.11. Let

$$a_k = \begin{cases} \frac{1}{2^{k+1}} & k \text{ odd} \\ \\ \frac{1}{2^{k-1}} & k \text{ even} \end{cases}$$

Then $\sum_{k\geq 0} a_k$ is the series

$$\frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \cdots$$

Note that $|a_{k+1}/a_k| = 1/8$ for even k and $= 2$ for k odd. So the limit $|a_{k+1}/a_k|$ does not exist and hence the ratio test does not apply. On the other hand, it is easy to check that $|a_k|^{1/k}| \to 1/2$ as $k \to \infty$ hence the root test implies the series converges absolutely. Note also that $|a_k| \leq 1/2^{k-1}$, so convergence of the series follows from comparision test as well.

One may wonder if there is a series which the ratio test shows it convergence but the root test gives no information. The answer is "no". Since if the limit $|a_{k+1}/a_k$ exists, then it equals the limit of $|a_k|^{1/k}$. We will outline a proof of this in the exericises. So together with Example 9.2.11, we see that the root test is a strictly stronger result. However, often the ratio test, if applicable, is more convenient.

There is another special type of series that deserves our attention. A series is **alternating** if the sign of consecutive terms alternate. By multiplying -1 to the terms, if necessary, we can assume the first term of the series is positive. Clearly, this operation will not affect whether the series is convergent or not. For alternating series, we have the following test of convergence.

Theorem 9.2.12. *Suppose $(a_i)_{i\geq 0}$ is a decreasing sequence of non-negative real numbers that converges to 0. Then the following alternating sequence converges.*

$$\sum_{k\geq 0}(-1)^k a_k = a_0 - a_1 + a_2 - \cdots$$

Sketch of proof. Using the assumptions $a_i \geq 0$ and (a_i) is decreasing, one shows that the sequence of partial sums of the alternating series satisfies:

$$s_1 \leq s_3 \leq \cdots \leq s_{2n+1} \leq \cdots \leq s_{2n} \leq \cdots \leq s_2 \leq s_0$$

In other words, the sequence of even partial sums is decreasing, bounded below by s_1 and the sequence of odd partial sums is increasing, bounded above by s_0. By completeness of real numbers, their limits, say L_0, L_1, respectively, exist. To show that whole sequence of partial sums converges, it suffices to show that $L_0 = L_1$. The triangle inequality implies,

$$|L_0 - L_1| \leq |L_0 - s_{2n}| + |s_{2n} - s_{2n+1}| + |s_{2n+1} - L_1|$$

Since $|s_{2n} - s_{2n+1}| = a_{2n+1}$ which, by assumption, tends to 0 as n tends to ∞. And by definition of limit, the other two terms tends to 0 as n tends to ∞ as well. Therefore, we conclude that $L_0 = L_1$ and the common value is the sum of the alternating sequence $\sum_{k\geq 1}(-1)^k a_k$. □

Example 9.2.13. The sequence $\frac{1}{k+1}$ is clearly positive, decreasing and converges to 0. Thus it follows from the alternate series test that the series

$$\sum_{k\geq 0}(-1)^k \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

converges. In fact, we will see that the sum is actually $\ln 2$ when we study power series expansions of functions.

Exercises

1. Show that an absolutely convergent series is a convergent series.

2. Find the sum of $\sum_{k\geq 1} \frac{1}{k^2+k}$.

9.3 Power Series

A **power series** in x is a series of the form

$$\sum_{n=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \tag{9.3}$$

We call the a_k's the **coefficeients**. Naively, a power series is an "infinte long" polynomial. And it represents a function of x on the domain of which it converges.

Example 9.3.1. The power series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$ is geometric with common ratio x. Hence, it converges to $1/(1-x)$ on $|x| < 1$. In other words, on $(-1, 1)$,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots \tag{9.4}$$

We should emphasis that the domain of $1/(1-x)$ is the set of all real numbers except 1, however, Equation 9.4 holds only when $|x| < 1$. For instance, when $x = 2$, the left side of (9.4) is $-1/2$ while the right side diverges.

More generally, a power series **centered at** x_0 is a series of the form

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{9.5}$$

So a series of the form (9.3) is a power series with center at the origin. The series (9.5), clearly, converges to a_0 at $x = x_0$. In general, we have the following fundamental result about the domain of convergence of power series

Theorem 9.3.2. *Given a power series $\sum_{k=0}^{\infty} a_k (x-x_0)^k$, there exists an $R \in [0, \infty]$, called the* **radius of convergence**, *such that the series*

1. *converges absolutely on $|x - x_0| < R$.*

2. *diverges on $|x - x_0| > R$.*

Moreover, one can show that $\limsup |a_k|^{1/k} = 1/R$ with the convention $1/0 = \infty$ and $1/\infty = 0$. And if $\lim |a_{k+1}/a_k|$ exists then it also equals $1/R$. Convergence of the series at $x = \pm R$ is more tricky as we shall see in the next couple examples.

Example 9.3.3. The series

$$\sum_{k\geq 0} \frac{x^k}{k!} = 1 + \frac{x}{2!} + \frac{x}{3!} + \cdots$$

has $a_k := 1/k!$ as the coefficient of the x^k term. Since

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{1}{(k+1)!}\frac{k!}{1} = \frac{1}{k+1} \longrightarrow 0$$

as $k \to \infty$. We conclude that the radius of convergence is $1/0 = \infty$. Thus the power series represents a function defined on the whole real line and as we shall see this function is actually e^x.

Example 9.3.4. Consider the series $\sum_{k\geq 0} k!x^k$. This time the coefficient of x^k is $a_k := k!$, hence

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} k+1 = \infty.$$

Thus the radius of convergence for this series is $1/\infty = 0$.

Example 9.3.5. Consider the power series $\sum_{k\geq 0} x^k/k$. Since $\lim |1/(k+1)/1/k| = \lim k/k+1 = 1$ hence the radius of convergence $1/1 = 1$. Thus from Theorem 9.3.2, we know that the series convergence absolutely on $(-1, 1)$. At $x = 1$, the series becomes the harmonic series which is divergent (see Example 9.2.2). On the other hand, at $x = -1$, the series converges by the alternate series test, thus the domain of convergence of the series is $[-1, 1)$.

Example 9.3.6. Consider the power series $\sum_{k\geq 0} x^k/k^2$. Again since

$$\lim \left|\frac{1/(k+1)^2}{1/k^2}\right| = \lim \frac{k^2}{(k+1)^2} \longrightarrow 1$$

the radius of convergence is 1 and since $\sum 1/k^2$ converges so the series converges at both $x = \pm 1$. Therefore the domain of convergence is $[-1, 1]$.

Example 9.3.7. Consider the power series centered at 1

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

It is easy to see that the radius of convergence is 1 by considering the limit of ratios of successive coefficients. So the domain of convergence is $(0,2)$ plus perhaps an endpoint or two. Compare with the harmonic series, we know that the series diverges to $+\infty$ at $x = 0$ and the alternating series test shows that it converges at $x = 2$. Thus the domain of convergence is $(0,2]$. In fact, the series represents $\ln x$ on $(0,2]$. See exercise.

Power series, and hence the functions represented by them, behave very nicely on the domain of convergence.

Theorem 9.3.8. *Suppose $\sum_{k=0}^{\infty}$ is a power series with radius of convergence $R > 0$. Then for any $0 < r < R$, the power series converges on $[-r,r]$ to a continuous function.*

The convergence is actually uniform on $[-r,r]$ which is a new concept needed in the proof as well. As continuity is a local property, it follows immediately from the theorem above that

Corollary 9.3.9. *The power series $\sum_{k=0}^{\infty} a_k x^k$ converges to a continous function on $(-R,R)$ where R is the radius of convergence.*

We should point out that a power series with a positive radius of convergence R need not converge uniformly on $(-R,R)$ (see exercise).

Not only that the function of which a power series converges to is continuous, in fact, it is both differentiable and integrable. Moreover, the integral and derivative of the function are represented by integrating and differentiating the power series term-by-term.

Lemma 9.3.10. *The power series*

$$\sum_{k=0}^{\infty} a_k x^k, \qquad \sum_{k=0}^{\infty} k a_k x^{k-1} \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

all have the same radius of convergence.

Theorem 9.3.11. *Suppose $\sum_{k=0}^{\infty}$ has radius of convergence $R > 0$ representing a function $f(x)$. Then for $|x| < R$,*

$$\int_0^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}.$$

Theorem 9.3.12. *Suppose $\sum_{k=0}^{\infty}$ has radius of convergence $R > 0$ representing a function $f(x)$. Then $f(x)$ is differentiable on $(-R,R)$ and*

$$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1} \qquad \text{for} |x| < R.$$

Theorem 9.3.13 (Abel's Theorem)**.** *Suppose $\sum_{k=0}^{\infty} a_k x^k$ has finite positive radius of convergence R and let f be the function that it represents. Then f is continuous at $x = R$ (resp. $x = -R$) if the series converges at $x = R$ (resp. $x = -R$).*

Exercise

1. Consider the power series $\sum_{k=0}^{\infty} 2^{-k} x^{3k}$. What is the radius of convergence?

2. Show that $\sum_{k \geq 1} x^k / 2^k$ converges to a continuous function on $(-2, 2)$ but the convergence is not uniform.

9.4 Taylor Expansions

As we have seen in the last section, a power series and hence is the function that it represents behaves extremely well on its domain of convergence, for instance the function is infinitely differentiable and its derivatives can be computed by differentiating the series term-by-term. In this section, we will study the basis of the class of functions that can be represented by power series.

The first thing is note is the following.

Theorem 9.4.1. *Suppose $f(x)$ is the function represented by a power series of the form $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ with a radius of convergence $R > 0$ then f is infinitely differentiable on $(x_0 - R, x_0 + R)$ and*

$$a_k = \frac{f'(x_0)}{k!}(x - x_0)^k$$

Proof. We have already seen that $f(x)$ is infinitely differentiable on $(x_0 - R, x_0 + R)$. Moreover, the derivatives of f can be found by differentiating the series term-by-term successively, for instance

$$\begin{aligned} f'(x) &= \Big(a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots\Big)' \\ &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots \end{aligned}$$

Thus $f'(x_0) = a_1$. Differentiate one more time, we get

$$\begin{aligned} f''(x) &= \Big(a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots\Big)' \\ &= 2a_2 + 3!a_3(x - x_0) + (4)(3)a_4(x - x_0)^2 + \cdots \end{aligned}$$

So $f''(x_0) = 2a_2$. One more differentiation and substituting x by x_0, we have $f'''(x_0) = 3!a_3$. Continue this way, one sees easily that $f^{(k)}(x_0) = k!a_k$. $\square$

We leave it as an exercise to the reader to provide a rigorous proof using induction. We call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \tag{9.6}$$

the **Taylor series** of $f(x)$ at x_0 and the polynomial

$$f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{9.7}$$

the **degree n Taylor polynomial** of $f(x)$ at x_0. When $x_0 = 0$, series (9.6) is also known as the **Maclaurin series** of $f(x)$.

An immediate corollary of Theorem 9.4.1 is the uniqueness of power series representation of functions.

Corollary 9.4.2. *Suppose a function $f(x)$ is represented by power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ and $\sum_{k=0}^{\infty} b_k(x-x_0)^k$ on an open interval containing x_0 then $a_k = b_k$ for all $k \geq 0$.*

Proof. By Theorem 9.4.1, both a_k and b_k equal $f^{(k)}(x_0)/k!$. □

Essentially, Theorem 9.4.1 says that if a function is represented by a power series on some open interval then the series must be its Taylor series at some point. However, as the following example indicates, a smooth function needs not be represented by its power series on the interval of convergence.

Example 9.4.3. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then one can show that (see exercise) $f^{(k)}(0) = 0$ for all $k \geq 0$ and so the Taylor series of $f(x)$ at 0 is identically zero and that only agrees with the value of $f(x)$ at $x = 0$ but nowhere else.

We say that a function $f(x)$ is **analytic at** x_0 if it is represented by a power series on an open interval containing x_0 and $f(x)$ is **analytic** if it is analytic at every point. A function analytic at a point is necessarily smooth at that point but not vice versa as shown by the previous example. So how can we tell whether a given function is analytic at a given point? The following result provides a way to answer this question.

Theorem 9.4.4. *Suppose f is defined on some open interval containing* 0*, and that the n-th derivative of f exists on I. Then for all $x \in I$*

$$R_n(x) = \frac{f^{(n)}(y)}{n!}x^n.$$

for some y between x and 0*. In particular, if the derivatives of f are uniformly bounded on I, i.e. if there exists a constant C such that $|f^{(n)}(x)| \leq C$ for all $x \in I$, then*

$$\lim_{n\to\infty} R_n(x) = 0.$$

There are other versions of Taylor's theorem involving integrals. We leave them in the exercises.

We are going to find the power series expansion of a few common analytic functions at various points.

Example 9.4.5. The geometric series

$$1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k$$

converges absolutely to the function $\dfrac{1}{1-x}$ on $(-1,1)$. Note that the function $1/(1-x)$ defines on $\mathbb{R}$ excepts at 1 but the series converges only on $(-1,1)$.

Example 9.4.6. Let $f(x) = e^x$. So $f^{(n)}(x) = e^x$ for all $n \geq 1$. So for any $M > 0$, $|f^{(n)}(x)| \leq e^M$ on the interval $(-M, M)$. By Theorem 9.4.4, e^x is analytic on the real line and is represented by the power series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (x \in \mathbb{R}).$$

Example 9.4.7. Let $f(x) = \sin x$, then

$$f^{(n)}(x) = \begin{cases} \cos x & n \equiv 1 \pmod 4 \\ -\sin x & n \equiv 2 \pmod 4 \\ -\cos x & n \equiv 3 \pmod 4 \\ \sin x & n \equiv 0 \pmod 4 \end{cases}$$

Hence

$$f^{(n)}(0) = \begin{cases} 1 & n \equiv 1 \pmod 4 \\ -1 & n \equiv 3 \pmod 4 \\ 0 & n \equiv 0,2 \pmod 4 \end{cases}$$

Thus the Maclaurin series for $\sin x$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \tag{9.8}$$

And since the derivatives of $\sin x$ are all bounded by 1, so $\sin x$ is represented by its Maclaurin series on $\mathbb{R}$. By Theorem, $\cos x$, the derivative of $\sin x$, is represented by the power series obtained by differentiating (9.8) term-by-term, so for $x \in \mathbb{R}$,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{9.9}$$

Example 9.4.8. Let us look at the function $f(x) = \ln(1+x)$ on $(-1, \infty)$. Taking derivatives, we have

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}, \quad \text{etc.}$$

Then an easy induction argument shows that

$$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}.$$

In particular, the Taylor series expansion of $f(x)$ at $x = 0$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots . \tag{9.10}$$

Another way of finding the Taylor series of $\ln(1+x)$ at $x = 0$ is to observe that $\dfrac{d\ln(1+x)}{dx} = \dfrac{1}{1+x}$ which according to Example 9.4.5 has

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots = \sum_{k=0}^{\infty} (-x)^k \end{aligned}$$

as its Taylor series expansion at $x = 0$. Thus by integrating the series term-by-term, we conclude that on $(-1, 1)$ the series

$$\int (1 - x + x^2 - x^3 \cdots)\, dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

represent that function $\ln(1+x) + k$ for some constant k. By evalutating both the series and the function at $x = 0$, we see that $k = 0$. Thus on $(-1, 1)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

which is precisely Equation (9.10).

Example 9.4.9. As we have seen from the examples above, substitution is a powerful technique in finding series expansions of functions. Let us illustrate this point again by finding the MacLaurin series (i.e. the Taylor series at $x = 0$) of $\arctan(x)$. First note that $(\arctan(x))' = 1/(1+x^2) = 1/(1-(-x^2))$ and so by Example 9.4.5,

$$1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots = 1 - x^2 + x^4 - x^6 + \cdots$$

represents the derivative of $\arctan(x)$ for all $|x^2| < 1$, i.e. for $|x| < 1$. Now by integrating the above series term-by-term, we see that on (-1,1),

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

represent $\arctan(x) + k$ for some constant k. Again k must be zero because both $\arctan(x)$ and the series above vanish at $x = 0$. Finally, note that the series converges (by the alternating test) at both end-points. Thus we conclude that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \tag{9.11}$$

on $[-1, 1]$. In particular,

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{9.12}$$

This give us a way of approximating π by the partial sums of a series.

Example 9.4.10. Let us give an example which illustrate how to radius of convergence may change after substitution. For instance, it follows from the previous example that

$$\begin{aligned} x \arctan(2x) &= x \left((2x) - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \cdots \right) \\ &= x \left(2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + \cdots \right) \\ &= \left(2x^2 - \frac{8x^4}{3} + \frac{32x^6}{5} - \frac{128x^8}{7} + \cdots \right). \end{aligned}$$

when $2x \in [-1, 1]$, i.e. when $|x| \leq 1/2$ and the radius of convergence of the last series is $1/2$.

Example 9.4.11. Let us compute the Taylor series of $\sin(x)$ at $\pi/2$. According to Taylor's theorem,

$$\sin(x) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(\pi/2)}{k!} \left(x - \frac{\pi}{2} \right)^k.$$

On the one hand,

$$\sin(\pi/2) = 1, \sin'(\pi/2) = \cos(\pi/2) = 0,$$
$$\sin''(\pi/2) = -\sin(\pi/2) = -1, \sin^{(3)}(\pi/2) = -\cos(\pi/2) = 0, \ldots$$

and the pattern repeats, so we have

$$\sin(x) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 + \cdots$$

On the other hand, using the trigonometric identity $\sin(x) \equiv \cos(x + \pi/2)$ and the Taylor series of $\cos(x)$ at 0, we also get

$$\sin(x) \equiv \cos\left(x + \frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{(2k)!}\left(x + \frac{\pi}{2}\right)^{2k}.$$

Example 9.4.12. Let us end the section by finding the MacLaurin series of $\frac{1}{(1-x)^2}$. On the one hand

$$\frac{1}{(1-x)^2)} = \left(\frac{1}{1-x}\right)^2 = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)$$

By "foiling" out the right side, one gets

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

On the other hand, one can avoid the "risky" algebra by observing

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right)\\ &= 1 + 2x + 3x^2 + \cdots\end{aligned}$$

Very cool, isn't it!

Index