Calculus: An intuitive approach

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Part I

Single Variable

Chapter 1

Differentiation

1.1 Motivation

We begin with the velocity-time graph of a smooth ride.

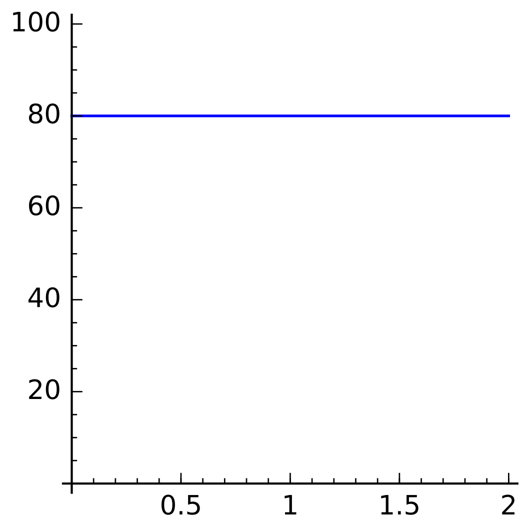


Figure 1.1: The velocity-time graph of a car.

It was boring for the driver since the car was traveling at a constant speed. In other words, the *acceleration*, i.e. the *rate of change of velocity* is zero at any time of this trip. Note that this fact is reflected by the slope of the graph (a horizontal line in this case) being zero at every point. What else about the trip can we tell from this graph? Well, the driver traveled 2 hours at the constant speed of 80 mph, so she traveled $80 \times 2 = 160$ miles. But can we "see" this information from the graph? Yes, it is the area of the rectangle underneath the graph!

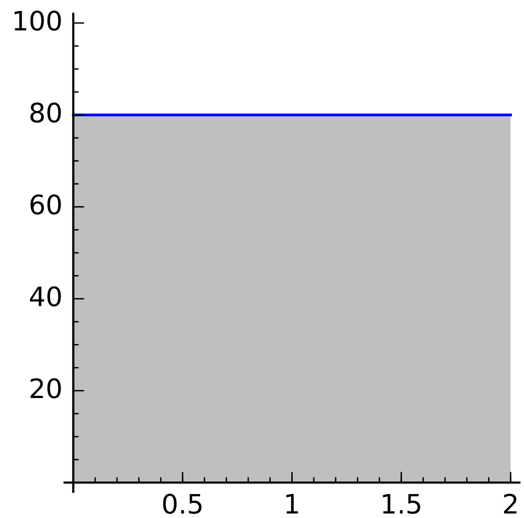


Figure 1.2: The shaded area represent the distance traveled by the car.

However, as you can expect the velocity-time graph of a trip in real traffic is much more complicate. In fact, even nice functions in mathematics, polynomials, trigonometric functions, exponential functions, etc, are still far more complex that a constant function... We surely need some tools to analyze them!

1.2 Derivative

In the previous section we have seen some reasons why one may be interested in finding (the slopes of) tangents. We now discuss how to find them. First, a working definition:

> The **derivative of a function f at a point a in the domain of f** is the slope (if exists) of the tangent to the graph f at the point $(a, f(a))$.

This definition assumes we know what a tangent is. In fact, it assumes a function, or rather its graph, can have at most one tangent at each point. We will give a precise definition of derivative in Section 1.6. For now let us think of the tangent of a graph at a point p as what the secant $\overline{pq}$ turns into when q approaches to p. We often refer the tangent of the graph of f at the point $(a, f(a))$ simply as the *tangent of f at a*.

A geogebra applet showing how the secants turn into the tangent.

The two most common notation for the derivative of f at a are

$$f'(a) \quad \text{and} \quad \frac{df}{dx}(a).$$

The first one is due to Lagrange, besides being compact, it suggests that the derivatives of f at various points are the values of another function namely f'. The second notation is due to Leibniz. It not only reminds us that a derivative is related to some sort of ratio but also reminds us that f is being considered as a function of x. So, for example, if f is considered as a function of z, then the second notation changes to $\frac{df}{dz}(a)$. If we think of a function f as how a dependent valuable y is relating to an independent valuable x, then we may also write its derivative at a as

$$\frac{dy}{dx}(a), \quad \left.\frac{dy}{dx}\right|_{x=a}, \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{(a,f(a))}.$$

Definition 1.2.1. We say that f is **differentiable** at a if $f'(a)$ exists and say that f is **differentiable** on a set D if f is differentiable at a for every $a \in D$. The function $a \mapsto f'(a)$ is called the **derivative** of f. We denote the derivative of $f(x)$ as $f'(x)$ or $\frac{df}{dx}$.

Intuitively, it is also clear that differentiability is a *local property*: that means if f and g agree near a, i.e. on an open interval containing a, then $f'(a) = g'(a)$ if either of them exists.

Example 1.2.2. Suppose $f(x) = mx + b$ is a linear function. The graph of f is the line $y = mx + b$ and the tangent at every point is the line itself. Hence $f'(x)$ is the constant function m. In particular, the derivative of a constant function ($m = 0$) is the zero function and the derivative of the identity function $f(x) = x$ is the constant function 1.

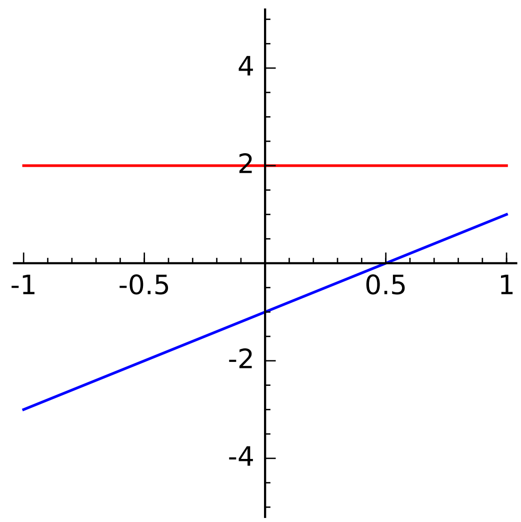


Figure 1.3: The graph of $2x - 1$ (blue) and its derivative (red).

Example 1.2.3. Consider the function $r(x) = 1/x$

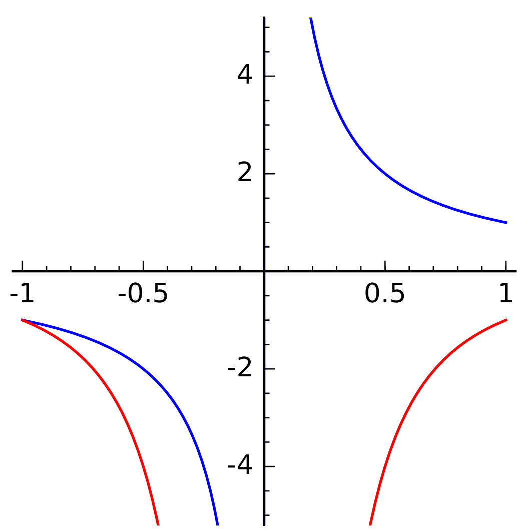


Figure 1.4: The graph of $\frac{1}{x}$ (blue) and its derivative (red)

The slope of the secant of two points on the graph is

$$\frac{r(b)-r(a)}{b-a}=\frac{1/b-1/a}{b-a}=-\frac{1}{ba}.$$

Letting $b \to a$, we see that the derivative of $1/x$ at the point a is $-1/a^2$. Since $a \neq 0$ is arbitrary, we conclude that $(1/x)' = -1/x^2$. Let us remark that in order for $b \to a$, we need a and b are of the same sign since the domain of $1/x$ does not include 0.

Example 1.2.4. Consider the absolute value function.

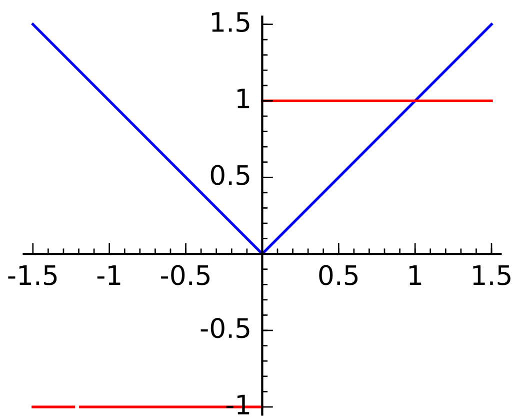


Figure 1.5: The graph of $|x|$ and its derivative

Clearly, the derivative of $|x|$ is 1 at every positive real and -1 at every negative real. The secant joining $(0,0)$ and $(a,|a|)$ is $y=x$ if $a>0$ and is $y=-x$ if $a<0$. They do not approach to the same line when a approaches 0. Therefore, $|x|$ is not differentiable at the origin. This is our first example of a function that is not differentiable at certain point. Roughly speaking, a graph has no tangent at "sharp corners".

Example 1.2.5. Let $f(x)=\sqrt{1-x^2}$. Even though we do not know $f'(x)$ explicitly yet, nonetheless intuitively $f'(0)$ should be 0 since the corresponding tangent is horizontal. Moreover, f' does not exists at $x=\pm 1$ since the corresponding tangents are vertical hence do not have a slope.

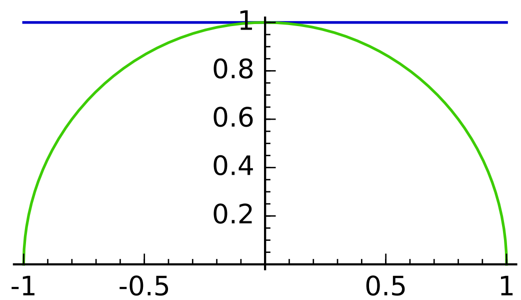


Figure 1.6: The graph of $\sqrt{1-x^2}$ and its tangent at $(0,1)$

1.3 Rules of Differentiation

So far we can only find the derivatives of a very small class of functions. To do more we need some practical ways of computing derivatives.

1.3.1 Sum and Product Rules

Theorem 1.3.1. *Suppose f, g are both differentiable at a then*

1. $(f+g)'(a) = f'(a) + g'(a)$

2. $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

The first statement tells us the derivative of a sum is the sum of the derivatives of the summands. Perhaps no surprise there. The second statement is called the **product rule**. It tells us how to compute the derivative of a product in terms of its factors and their derivatives. This is probably the "first surprise" in calculus since $(fg)'(a) \neq f'(a)g'(a)$ in general[1]. Let us note that if either f or g is not differentiable at a, their sum and product might still be differentiable at a. Let us illustrate this point by a couple examples:

Example 1.3.2. Let $f(x) = |x|$ and $g(x) = -|x|$. As we have seen both f and g fail to be differentiable at 0. However, $(f+g)(x) \equiv 0$ hence differentiable everywhere and at 0 in particular.

Example 1.3.3. Let $f(x) = |x|$ and $g(x) = x$. Then $f(x)$ is not differentiable at 0

[1] But this is a nice surprise. The theory of Calculus would be far less interesting if otherwise.

hence the right-hand side of (2) does not make sense at $x = 0$. However,

$$(fg)(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

thus the secants on either side of 0 approaches to the line $y = 0$ therefore $(fg)'(0) = 0$.

Let us also illustrate how to compute the derivatives of polynomials and power functions using the two rules in Theorem 1.3.1.

Example 1.3.4. Since the derivative of a constant function is the zero function, the product rule implies:

$$(cf(x))' = (c)'f(x) + 2f(x)' = 0(f(x)) + cf'(x) = cf'(x).$$

Thus we can "take the constant out" when taking the derivative.

Example 1.3.5. Since $x' = 1$, it follows from the product rule that

$$(x^2)' = x'x + xx' = x + x = 2x.$$

From this we can deduce

$$(x^3)' = (x^2)'x + x^2(x)' = 2x(x) + x^2(1) = 3x^2.$$

If you are familiar with mathematical induction, it is an easy exercise to show that $(x^n)' = nx^{n-1}$ for any natural number n. Since single variable polynomials can be constructed from the variable and constants via addition and multiplication. We can compute the derivative of polynomials rather easily. For example,

$$\begin{aligned} (2x^3 - x^2 + 5x + 8)' &= (2x^3)' + (-x^2)' + (5x)' + (8)' \\ &= 2(x^3)' - (x^2)' + 5(x)' + (8)' \\ &= 2(3x^2) - 2x + 5 + 0 \\ &= 6x^2 - 2x + 5. \end{aligned}$$

In Example 1.2.3, we show that the $r(x) = 1/x$ is differentiable. Using the product rule and induction, one can show that $(x^n)' = nx^{n-1}$ holds even for negative n. In fact, the equality still holds for any real exponent.

Theorem 1.3.6 (Power Rule)**.** *For any $r \in \mathbb{R}$, $(x^r)' = rx^{r-1}$.*

We will prove the **power rule** in Section 1.6. As an example, take $r = 1/2$ then the power rule asserts that

$$(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Example 1.3.7. Let us compute the derivative of $f(x) = \sqrt{x-2}$ with respect to x. At the first sight, it seems that none of the rules that we have discussed can use to solve the problem. However, we will show that it is not the case and in fact offer two "different" ways of solving the problem. First by examining the graph of f, we see that it is differentiable everywhere except at $x = 2$.

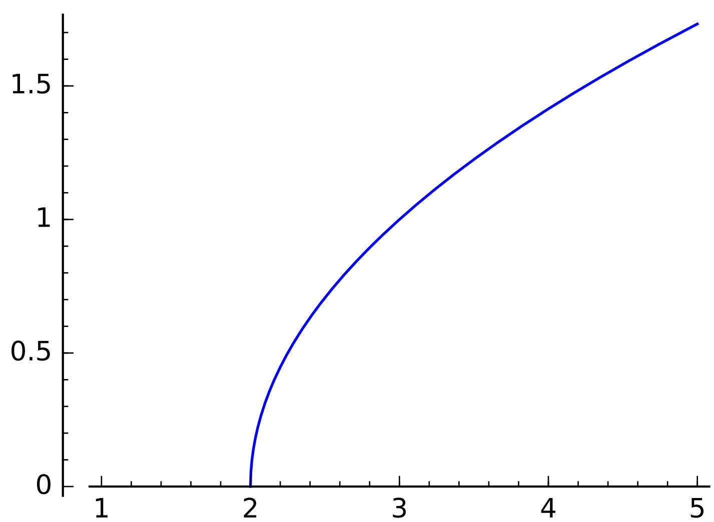


Figure 1.7: The graph of $\sqrt{x-2}$

And so for $x \neq 2$, $f'(x)$ can be computed by differentiating $(f \cdot f)(x) = x + 2$ using the product rule. That gives $2f'(x)f(x) = 1$ and so

$$f'(x) = \frac{1}{2f(x)} = \frac{1}{2\sqrt{x-2}}.$$

The second way is more geometrical.

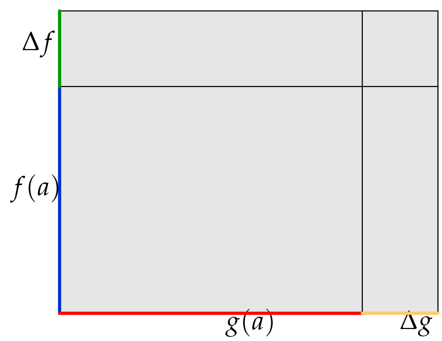


Figure 1.9: A digram for the product rule

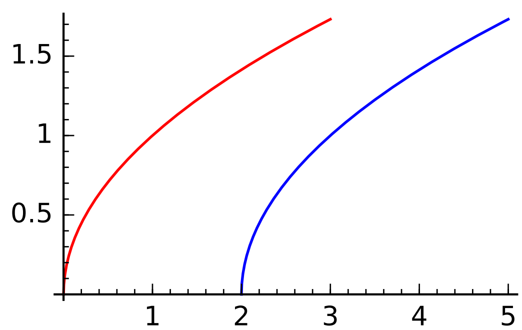


Figure 1.8: The graph of $\sqrt{x}$ and the graph of $\sqrt{x-2}$

Since the graph if $\sqrt{x-2}$ is a horizontal shift of the graph of $\sqrt{x}$, the same relation holds for their derivatives. Hence, the graph of the derivative of $\sqrt{x-2}$ must be just the shift of the graph of $(\sqrt{x})' = 1/(2\sqrt{x})$ (by power rule) to the right by 2. Thus $f'(x)$ must be $1/(2\sqrt{x-2})$.

We shall proof the product rule in Section 1.6. Here we will give an idea why it should take that form by a digram (Figure 1.9). The rectangle represents the quantity $fg + \Delta(fg)$ when x is changed by Δx at $x = a$ [2]. From the figure we see that

$$(fg)(a) + \Delta(fg) = f(a)g(a) + f(a)\Delta g + g(a)\Delta f + \Delta f \Delta g$$
$$\Delta(fg) = f(a)\Delta g + g(a)\Delta f + \Delta f \Delta g.$$

[2] Δfg maybe negative, but the reader should get the idea.

As $\Delta x \to 0$, the left-side of the equation above tends to $(fg)'(a)$ and on the right-side $\Delta f/\Delta x \to f'(a)$, $\Delta g/\Delta x \to g'(a)$ and $(\Delta f)(\Delta g)/\Delta x$ tends $f'(a)(0) = 0 = (0)g'(a)$. Thus, we should expect,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

1.3.2 Exercises

1. Show that $(x^{\frac{1}{q}})' = \frac{1}{q}x^{\frac{1}{q}-1}$ for any integer $q \neq 0$. Deduce that $(x^r)' = rx^{r-1}$ for any real number r.

2. Justify that $\sqrt{x}$ is differentiable for all $x > 0$ by computing the limit $\lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

3. Consider the following two functions defined on $[0,4]$

$$f(x) = \begin{cases} x & 0 \le x \le 2 \\ 4-x & 2 \le x \le 4 \end{cases}. \qquad g(x) = x-2.$$

   (a) Graph $f(x)$ and $g(x)$.

   (b) Let $h = fg$. Compute $h'(1)$ and $h'(3)$.

   (c) The function h is indeed differentiable at $x = 2$. Find the value of $h'(2)$. Why does it not contradict the product rule?

1.3.3 The Chain Rule

Even with the rules that we have developed, we can justify the differentiability for only a fairly small class of functions. The situation changes once we established the **Chain Rule**. It deals with differentiability of compositions of functions. It is the key result for justifying differentiability of functions and finding their derivatives.

Example 1.3.8. Before stating the chain rule, let us consider

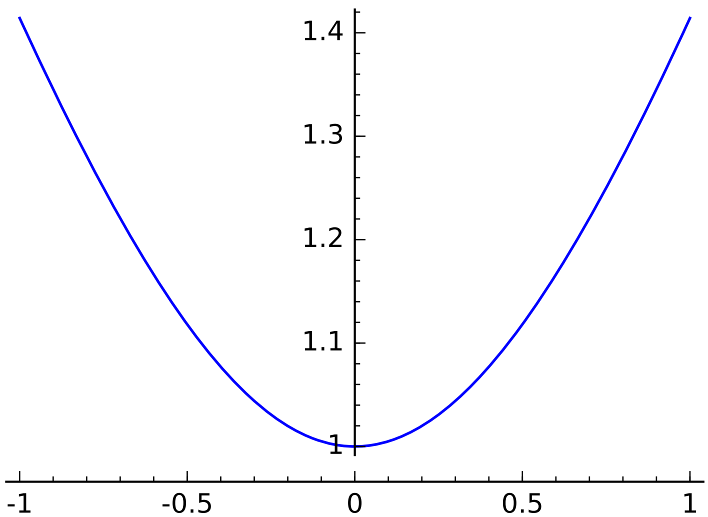


Figure 1.10: The graph of $\sqrt{x^2+1}$

From its graph, it is clear that $h(x) = \sqrt{x^2+1}$ is differentiable (at least on the part that it is shown). However, we cannot justify this fact by the rules that we have so far. Note that $\sqrt{x^2+1}$ is the composition $g \circ f(x)$ where $g(x) = \sqrt{x}$ and $f(x) = x^2+1$ both of them we know are differentiable and can compute their derivatives. So what we need is a result about differentiability that deals with composition of functions and that is exactly what the chain rule is about.

On the other hand, if we accept the fact that $h(x)$ is differentiable by the graphical justification, then we do can compute its derivative from the rules that we have so far. This is because $h^2(x) = x^2+1$ and so by the product rule $2h(x)h'(x) = 2x$ and hence $h'(x) = x/h(x) = x/\sqrt{x^2+1}$.

Theorem 1.3.9 (Chain Rule)**.** *Suppose f is a function differentiable at a and g is a function differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g(f'(a))f'(a) \tag{1.1}$$

We will give a proof of the chain rule in Section 1.6. Just like the previous two rules, the chain rule is not applicable if either $g'(f(a))$ or $f'(a)$ does not exist (see exercises for examples).

For computation, Equation 1.1 is what we need. It expresses the derivative of a composition, in terms of its components and their derivatives.

As an application, let us deduce the **quotient rule** from the product rule and the chain rule. Suppose $g(x)$ is differentiable at a and $g(a) \neq 0$. Since

$r(x) = 1/x$ is differentiable (Example 1.2.3). According to the chain rule, the function $(r \circ g)(x) = 1/g(x)$ is differentiable at a, moreover

$$\left(\frac{1}{g}\right)'(a) = (r \circ g)'(a) = r'(g(a))g'(a) = -\frac{g'(a)}{g^2(a)}$$

Now suppose f is also differentiable at a, it then follows from the product rule that

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\left(\frac{1}{g}\right)(a) \\ &= -\frac{f(a)g'(a)}{g^2(a)} + \frac{f'(a)}{g(a)} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.\end{aligned}$$

This is the **quotient rule**.

Suppose f is differentiable at a and has an inverse near a. Since the graph of f^{-1} is obtained by reflecting the graph of f along the line $y = x$, intuitively if f has a tangent at $(a, f(a))$, then f^{-1} will have a tangent at $(f(a), a)$. In fact, it would be the line obtained by reflecting the tangent of f at a along the line $y = x$ hence its slope of would be the reciprocal of the slope of the tangent of f at a. Indeed, we have

Theorem 1.3.10. *Suppose f has a nonzero derivative at a and has an inverse near a. Then f^{-1} is differentiable at $f(a)$, moreover,*

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}. \tag{1.2}$$

We will give a proof of this theorem in Section 1.6. The hard part is again to show that f^{-1} is differentiable at $f(a)$. Once this fact is established, then (1.2) is a simple consequence of the chain rule. Since $f^{-1}(f(x)) = x$, differentiate both sides with respect to x and evaluate at a yields,

$$(f^{-1})'(f(a))f'(a) = 1.$$

We obtain (1.2) on dividing both sides by $f'(a) \neq 0$.

1.3.4 Exercises

1. Let $f(x) = \begin{cases} 1 + x & x < 0; \\ 1 + \dfrac{x}{2} & x \geq 0. \end{cases}$ and $g(x) = \begin{cases} x - 1 & x < 1; \\ 2x - 2 & x \geq 1. \end{cases}$

(a) Is $f(x)$ differentiable at $x = 0$?

(b) Is $g(x)$ differentiable at $x = 1$?

(c) Is $g \circ f(x)$ differentiable at $x = 0$?

2. Let $f(x) = \sqrt{1 - x^2}$. Does the derivative of $(f \circ f)(x)$ at $x = 0$ exist?

1.4 Derivatives of Elementary Functions

1.4.1 Trigonometric Functions

To compute the derivatives of trigonometric functions, all we need to know is the derivative of the sine function since other trigonometric functions, e.g. cosine, are just simple transformation of sine. And if you sketch the graph of sine and the graph of its derivative by estimating the slopes of the tangents at various points, you will get a graph that looks very much like the following

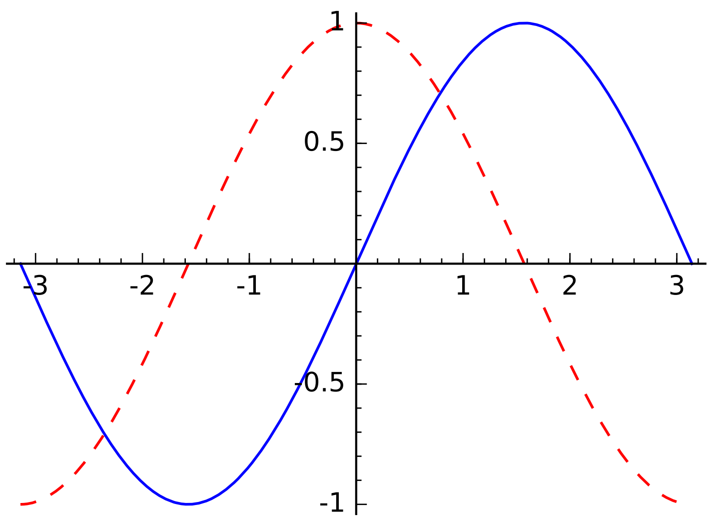


Figure 1.11: The graph of sine (blue) and the sketch of its derivative (red)

So

$$\frac{d}{dx} \sin x = \cos x \tag{1.3}$$

would be a reasonable guess. We will vindicate this intuition in Section 1.6. Since $\cos x$ is the shift of $\sin x$, more precisely, $\cos x \equiv \sin(x + \pi/2)$, the derivative of cosine is the shift, by the same among, of the derivative of sine. That means

$$\cos' x = \cos(x + \pi/2) \equiv -\sin x \tag{1.4}$$

Once we know the derivative of sine and cosine, the derivatives of the other the trigonometric functions follows from the quotient rule.

1.4.2 Exercises

1. Show that $\dfrac{d}{dx} \tan x = \sec^2 x$.
2. Find the derivative of the other trigonometric functions.

1.4.3 Exponential and Logarithmic Functions

Our goal is to find the derivatives of exponential and logarithmic functions. First, consider an exponential function b^x ($b > 0$). Intuitively, b^x is differentiable everywhere since its graph has no sharp corners and no vertical tangents.

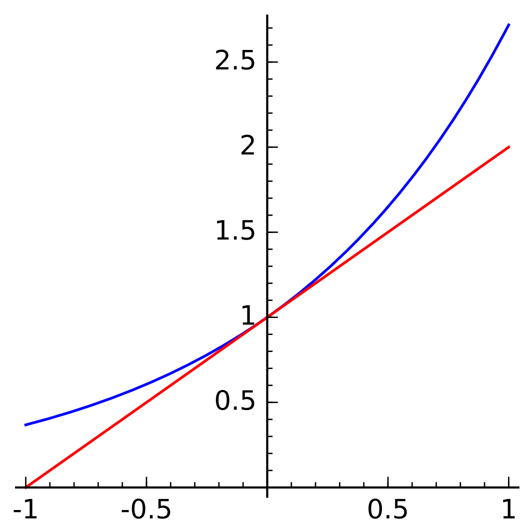


Figure 1.12: The graph of e^x and its tangent at $x = 0$

We will establish this fact in Section 1.6. Meanwhile, note that

$$\frac{b^{x+h} - b^x}{h} = b^x \left(\frac{b^h - 1}{h} \right) \tag{1.5}$$

so by letting $h \to 0$ in (1.5), we conclude that

$$\frac{d}{dx}b^x = b^x \left(\frac{d}{dx}b^x \bigg|_{x=0} \right). \tag{1.6}$$

This reveals an important feature of exponential function:

The derivative of an exponential function is a multiple of itself. Moreover the multiple is precisely the slope of its tangent at $x = 0$.

So it is perfectly natural to ask for which b will make that slope equal 1? In other word, which number b will the derivative of b^x to be itself? From Equation (1.5), we see that it requires $b^h - 1 \approx h$ when $h \approx 0$. In particular, it requires $b^{1/n} \approx 1 + 1/n$, in other words $b \approx (1 + 1/n)^n$, for large n. This prompts us to define the base of this special exponential function to be

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n. \tag{1.7}$$

We will show that the limit above exists in Section 1.6 and denote it by e. We call the function $x \mapsto e^x$ (also written as $\exp(x)$) the **exponential function**. Certainly, the whole point of making such a definition is to guarantee

$$\frac{d}{dx}e^x = e^x. \tag{1.8}$$

Let $\ln x$ be the inverse function of e^x (i.e. the logarithm with base e). We call it the **natural logarithm**. According to (1.2) and (1.8), $(\ln x)' = 1/x$. However, to illustrate the proof Theorem 1.3.10 and the advantage of Leibniz notation, we will derive $(\ln x)' = 1/x$ directly. Let $y = \ln x$, so $x = e^y$. Differentiate both sides with respect to x, we get

$$1 = \frac{dx}{dx} = \frac{d}{dx}e^y = \frac{d}{dy}e^y \frac{dy}{dx}$$

according to the chain rule. Thus,

$$\frac{d}{dx}\ln x = \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

1.4.4 Exercises

1. Assuming that the limit in (1.7) exists, show that $\dfrac{b^h - 1}{h} \to \ln b$ when $h \to 0$ and deduce that

$$\frac{d}{dx}b^x = (\ln b)b^x.$$

2. Use the chain rule to prove again that $(b^x)' = (\ln b)b^x$. (Hint: $b^x = e^{x \ln b}$)

3. Show that $\dfrac{d}{dx} \log_b x = \dfrac{1}{x \ln b}$. (Hint: $\log_b x = (\ln x)/(\ln b)$.)

4. Verify by the chain rule that $(ln(f))' = \dfrac{f'}{f}$ if both sides exist. (This is called the logarithmic derivative of f.)

5. For functions $f(x)$ and $g(x)$, the symbol $g(x)^{f(x)}$ means the function $e^{f(x) \ln g(x)}$ (also written as $\exp(f(x) \ln g(x))$. The domain of this function is the intersection of the domain of f and the set on which $g > 0$.

   Express the derivative the of g^f in terms of the derivatives of f and g.

1.4.5 Inverse Trigonometric Functions

The trigonometric functions are not 1-to-1 so by inverse trigonometric functions we actually mean the inverse functions of the restriction of trigonometric functions to various domains. For example, there are infinitely many numbers whose sine is $1/2$. By $\arcsin 1/2$ we mean the one in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So $\arcsin^{-1}(1/2) = \pi/6$.

Example 1.4.1. Let us compute the derivative of $\arcsin x$. We use the same technique as in computing the derivative of $\ln x$. Let $y = \arcsin x$, then $\sin y = x$ and so

$$1 = \frac{dx}{dx} = \frac{d \sin y}{dx} = \frac{d \sin y}{dy} \frac{dy}{dx} = \cos y \frac{dy}{dx}$$

Thus $dy/dx = 1/\cos y = \sec y$. To express dy/dx back in terms of x, we get a hint from the following diagram

insert a triangle here.

Thus $\sec y = 1/\sqrt{1 - x^2}$. Another way to get this is to use the identity

$$1 \equiv \cos^2 y + \sin^2 y = \cos^2 y + x^2,$$

Thus $\cos y = \pm\sqrt{1 - x^2}$. By examine the graph of $\arcsin x$, we see that the slopes of its tangents are non-negative hence we should take the positive root.

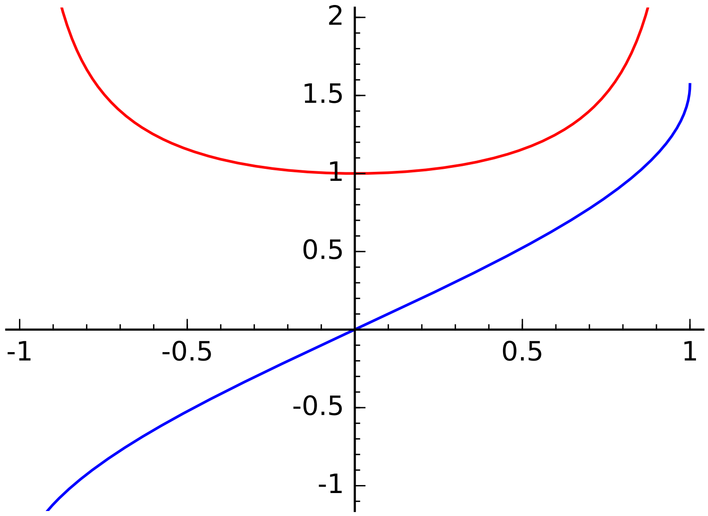


Figure 1.13: arcsin x and its derivative

In summary, arcsin x maps $[-1,1]$ to $[-\frac{\pi}{2},\frac{\pi}{2}]$ and

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$

The graph of cosine restricted to $[0,\pi]$ is a horizontal shift of the graph of sine restricted to $[-\frac{\pi}{2},\frac{\pi}{2}]$. Thus the graph of arccos x is a vertical shift of the graph of $-\arcsin x$ and hence the same derivatives (since their tangents at the corresponding points are parallel). Therefore, we conclude that arccos x maps $[-1,1]$ to $[0,\pi]$ and

$$\frac{d}{dx}\arccos x = -\frac{d}{dx}\arcsin x = -\frac{1}{\sqrt{1-x^2}}.$$

Exercise

1. Show that the derivative of arctan x is $\dfrac{1}{1+x^2}$.

1.5 Implicit Differentiation

So far we have only studied tangents of curves that are explicitly given as the graph of functions. However, there are curves, for example circles, in the xy-plane to which tangents makes sense but are not graphs of functions of either x

or y. In these cases, we can still find the tangents to these curves at various point by treating one of the variable as a function of the other locally. Let $F(x, y)$ be a function of x and y. We say that the relation $F(x, y) = 0$ defines y **implicitly as a function of x near a point** (a, b) if there is a function $y(x)$ defined on an open interval I containing a such that $y(a) = b$ and $F(x, y(x)) = 0$ for all x in I.

Example 1.5.1. The relation $x^2 + y^2 = 25$ defines y as $\sqrt{25 - x^2}$ near $(3, 4)$. The same relation defines y as $-\sqrt{25 - x^2}$ near $(3, -4)$. Let us find the tangent to this circle at the point $(3, -4)$ without explicitly solving y as a function of x. To do this, we differentiate both sides of the equation keeping in mind that y is a function of x.

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}25 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ x + y\frac{dy}{dx} &= 0. \end{aligned}$$

So $3 + (-4)\left.\frac{dy}{dx}\right|_{(3,4)} = 0$, therefore $\left.\frac{dy}{dx}\right|_{(3,4)} = \frac{3}{4}$. Thus the tangent to the circle at $(3, -4)$ is $3x - 4y = 25$.

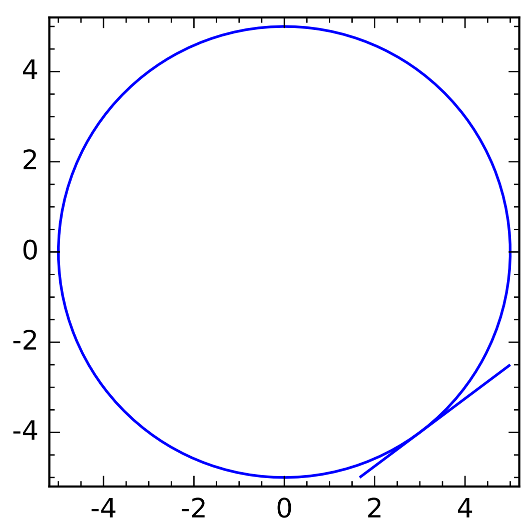


Figure 1.14: The circle $x^2 + y^2 = 25$ and its tangent at $(3, -4)$

Example 1.5.2. From the picture, it is clear that the curve $y^2 = x + x^3$ defines x as a function of y near the origin. Although, it is not easy to express x as a function of y explicitly. Note also that near the origin y is not a function of x.

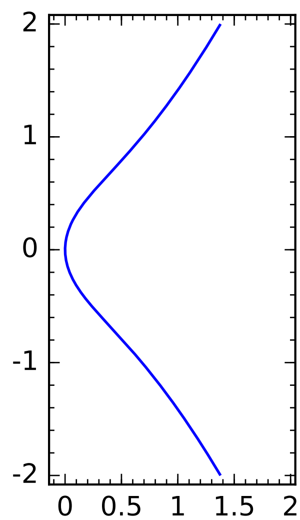


Figure 1.15: The curve $y^2 = x + x^3$

1.6 Continuity and Limits

In this section, we show how to put the results that we have encountered so far on solid grounds via the concept of continuity. We will give either proofs or references to those "black boxes" that we have being used and some that we are going to use.

1.6.1 Continuous Functions

What means by a function f is *continuous*? Roughly speaking, that means you can draw the graph of f without ever lifting your pen off the paper. But there are a couple problems:

1. It is not very useful in showing that properties of continuous functions (e.g. compositions of continuous functions is continuous).

2. some functions you simply cannot "draw its graph" (try drawing the graph of $x \mapsto \sin(1/x)$ near 0).

So what makes one can draw the graph without lifting one's pen? Well, one moment of reflection should convince you that: *A function f is continuous at a point x_0 in its domain if the value $f(x)$ is near the value $f(x_0)$ whenever x is near x_0.* If we use $A \simeq B$ to denote the A is "near" (or "close to") B then we can rephrase what we have just said symbolically as

$$x \simeq x_0 \implies f(x) \simeq f(x_0) \tag{1.9}$$

In other words, f is continuous at x_0 means the condition $f(x)$ is "closed to" $f(x_0)$ is guaranteed by the condition x is "close to" x_0. So if we can make precisely what means by "nearby" then we can make precise what means by continuous. It is at this point different branches of mathematics takes different approaches. For example, the approach taken in *Non-standard analysis* is by making the concept of *infinitesimal* precise. This is arguably the historic approach of Calculus taken by Newton and Leibniz and was put onto a solid ground of logic by A.Robinson in 1960. In this approach, the implication in (1.9) as $f(x)$ is infinitesimally close to $f(x)$ whenever x is infinitesimally close to x_0 (reference J.Kiesler). In topology, the approach is by abstracting the properties of "being nearby". A function f is continuous at a point x_0 if $f^{-1}(U)$ is a neighborhood of x_0 whenever U is a neighborhood of $f(x_0)$. For us, we will take up the nowadays "standard" metric approach: f is continuous at x_0 if the distance of $f(x)$ and $f(x_0)$ can be *as small as one pleases* provided that the distance of x and x_0 is *small enough*. This leads to:

Definition 1.6.1. A real-valued function f is continuous at a point $x_0 \in \operatorname{dom} f$ if for any $\varepsilon > 0$, there exists a $\delta := \delta(\varepsilon, x_0) > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. A function is **continuous** on a subset S of its domain if it is continuous at every point of S.

Proposition 1.6.2. *Suppose f and g are continuous a a, then $f + g$ and fg are continuous at a also.*

Proposition 1.6.3. *Suppose f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a. Thus a composition of continuous functions is continuous.*

Theorem 1.6.4 (Intermediate Value Theorem)**.** *Suppose f is continuous on $[a, b]$ and $f(a)f(b) \leq 0$ then $f(c) = 0$ for some $c \in [a, b]$.*

Corollary 1.6.5. *Suppose f is continuous on $[a, b]$ and d is a real number between $f(a)$ and $f(b)$ then there is a $c \in [a, b]$ such that $f(c) = d$.*

Proof. Apply Theorem 1.6.4 to the continuous function $g(x) := f(x) - d$. □

1.6.2 Differentiable functions

We adopt Carathéodory's formulation of the derivative.

Definition 1.6.6. Suppose f is a function defined on $X \subseteq \mathbb{R}$ and a is an interior point of X. We say that f is **differentiable** at a, if there exists a function φ continuous at a such that for every $x \in X$,

$$f(x) - f(a) = \varphi(x)(x-a).$$

Moreover, the value $\varphi(a)$ is called the **derivative** of $f(x)$ at a.

We give a proof of the product rule.

Proof. Suppose f and g are differentiable at $x = a$, that mean there exist functions φ, ξ continuous at a such that $f(x) - f(a) = \varphi(x)(x-a)$ and $g(x) - g(a) = \xi(x)(x-a)$ on some open interval I containing a. So on I,

$$\begin{aligned} f(x)g(x) - f(a)g(a) &= f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a) \\ &= g(x)(f(x) - f(a)) + f(a)(g(x) - g(a)) \\ &= g(x)\varphi(x)(x-a) + f(a)\xi(x)(x-a) \\ &= (g(x)\varphi(x) + f(a)\xi(x))(x-a). \end{aligned}$$

Since $g(x), \varphi, \xi$ are all continuous at a, so is $g(x)\varphi(x) + f(a)\xi(x)$. Hence the last equation shows that $(fg)(x)$ is differentiable at $x = a$ with derivative $g(a)\varphi(a) + f(a)\xi(a) = f'(a)g(a) + f(a)g'(a)$. □

1.6.3 Trigonometric functions

Let us establish the fact $(\sin x)' = \cos x$. First we need the following limit.

Proposition 1.6.7. $\dfrac{\sin(h)}{h} \to 1$ *as* $h \to 0$.

Proof. For small positive h, $\sin(h) < h < \tan(h)$. Thus

$$\cos(h) < \frac{\sin(h)}{h} < 1.$$

Since $\cos(h) \to 1$ and so $\sin(h)/h \to 1$ as $h \to 0$ from the right. By symmetry a similar argument shows that $\sin(h)/h \to 1$ as $h \to 0$ from the left as well. This completes the proof. □

Theorem 1.6.8. $\dfrac{d\sin(x)}{dx} = \cos(x)$.

Proof. First

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sin(x)(\cos(h)-1)}{h} + \frac{\sin(h)}{h}\cos x.$$

We need to show that the expression above tends to $\cos(x)$ as $h \to 0$. We know that $\sin(h)/h \to 1$ by Proposition 1.6.7, so it remains to show that $(\cos(h) - 1)/h \to 0$ as $h \to 0$. To see this, note that as $h \to 0$,

$$\frac{\cos(h)-1}{h} = \frac{\cos^2(h)-1}{h(\cos(h)+1)} = -\frac{\sin(h)}{h}\frac{\sin(h)}{\cos(h)+1} \to 0$$

because $\sin(h)/(\cos(h)+1) \to 0$ and again $\sin(h)/h \to 1$ as $h \to 0$. □

1.6.4 A definition of e

We have argued that the exponential function $x \mapsto e^x = \exp(x)$ is differentiable provided that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists. We define e as this limit. Here we justify its existence. Fix an arbitrary natural number $n \geq 2$, according to the binomial theorem:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k = 1+1+\sum_{k=2}^{n}\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k \\ &= 2+\sum_{k=2}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \\ &< 2+\sum_{k=2}^{n}\frac{1}{k!} \leq 2+\sum_{k=1}^{n}\frac{1}{2^k} < 3.\end{aligned}$$

Note that the k-th term ($k \geq 2$) of the sum

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

increases as n increases. Moreover, the number of terms in the sum also increases as n increases. Thus $(1+1/n)^n$ is increasing and bounded above by 3. Therefore, the limit exists (and is less than 3).

Chapter 2

The Mean Value Theorem and its Applications

We study some applications of differential calculus that we have developed in the previous chapter.

2.1 The Mean Value Theorem

The **Mean Value Theorem** (MVT) is a key theorem in single valuable calculus.

Theorem 2.1.1 (Mean Value Theorem)**.** *Suppose f is a function continuous on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

In other words, if f is continuous on $[a, b]$ and differentiable on (a, b), then the tangent of f at some point $c \in (a, b)$ is parallel to the secant of f connecting the two end points of its graph.

Example 2.1.2. Let $f(x)$ be the restriction of $|x|$ to $[-1, 1]$. Then $f(-1) = 1 = f(1)$. However, the derivative of f is either 1 or -1, so there is no $c \in (-1, 1)$ such that $f'(c) = 0$. This shows that the assumption that the function is differentiable on (a, b) in the MVT is essential.

Example 2.1.3. Consider the function on $[-1, 1]$ defined by

$$f(x) = \begin{cases} x & -1 < x < 1 \\ 0 & x = -1, 1 \end{cases}$$

Clearly $f'(x) = 1$ for all $x \in (-1, 1)$, so there is no $c \in (-1, 1)$ such that

$$f'(c)(1 - (-1)) = 2f'(c) = f(1) - f(-1) = 0.$$

This shows that the assumption that f is continuous on $[a, b]$ in the MVT is also essential.

To proof the mean value theorem, we first establish a special case.

Theorem 2.1.4 (Rolle's Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Moreover, if $f(a) = 0 = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. The theorem is trivial if $f \equiv 0$ on $[a, b]$. So suppose f is not identically zero on $[a, b]$. Since $f(a) = 0 = f(b)$, this non-zero extreme value must be attained by some $c \in (a, b)$ (Theorem 2.2.2). Therefore, c must be a local extremum and since $f'(c)$ exists, therefore it must be 0 by the critical point theorem (Theorem 2.2.5). □

Proof of the mean value theorem. Let $s(x)$ be the linear function that defines the secant connecting the two end points of the graph of $f(x)$. Then the difference $g(x) := f(x) - s(x)$ satisfies the assumption of Rolle's Theorem. Therefore, there exists $c \in (a, b)$ such that

$$0 = g'(c) = f'(c) - s'(c) = f'(c) - m$$

where $m = \dfrac{f(b) - f(a)}{b - a}$ is the slope of the secant. □

Definition 2.1.5*.* Suppose f is defined on an interval I. We say that f is **increasing** (resp. **decreasing**) **on** I if $f(x_1) \le f(x_2)$ (resp. $f(x_1) \ge f(x_2)$) whenever $x_1, x_2 \in I$ and $x_1 \le x_2$.

Proposition 2.1.6. *Suppose f is continuous on an interval I and $f' \ge 0$ on the interior of I then f is increasing on I.*

Proof. For any $x_1, x_2 \in I$ with $x_1 < x_2$. Apply the MVT on the restriction of f to $[x_1, x_2]$, we conclude that there is some $c \in [x_1, x_2]$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \ge 0,$$

since $f'(c) \ge 0$ by the assumption on f'. □

Applying the argument above to $-f$, we conclude that if f is continuous on I with $f' \le 0$ on the interior of I, then f is decreasing on I. Therefore, if $f' \equiv 0$ on the interior of I, then f is both increasing and decreasing on I, hence,

Corollary 2.1.7. *Suppose f is differentiable on an interval I and $f' \equiv 0$ on the interior of I. Then f is constant on I.*

More generally,

Corollary 2.1.8. *Suppose f is defined on an open subset of $\mathbb{R}$ and its derivative is constantly 0, then f is locally constant.*

2.1.1 Exercises

1. (**Racetrack Principle**) Suppose f and g both define on a and $f'(x) > g'(x)$ (resp. $\geq$) for all $x > a$. Show that $f(x) > g(x)$ (resp. $\geq$) for all $x > a$.

2.2 Minima and Maxima of Functions

It is easy to see why one maybe interested in finding maxima and minima of a function. For example, one may want to maximize profits and minimize cost. So our first application of differential calculus is the study extrema of functions.

Definition 2.2.1. A point x_0 is a **global maximum** (resp. **global minimum**) of a function f if $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all x in the domain of f. A **global extremum** of f is either a global maximum or a global minimum of f.

In other words, a global maximum (minimum) of f is a point in its domain where f attains its maximum (minimum) value. The maximum (minimum) value is unique however that can certainly be attained at different points, for example, let $f(x)$ be the absolute value of x defined on $I = [-1, 1]$. Then the maximum value of f is 1 and are attained at both $x = -1$ and $x = 1$. So ± 1 are the global maxima of f. Certainly, a function needs not have a maxima or minima on its domain. For examples,

- The function $1/(x+1)$ on $[0, \infty)$ have 1 as its maximum value and is attained at $x = 0$. But it has no minimum value.
- The function $\tan(x)$ on $(-\pi/2, \pi/2)$ has neither global maxima nor global minima.
- The function f on $[-1, 1]$ defined by

$$f(x) = \begin{cases} 1/x & x \neq 0; \\ 0 & x = 0. \end{cases}$$

  has neither global maxima nor global minima.

However, a continuous function on a closed bounded interval must attains its maximum and minimum values:

Theorem 2.2.2 (Extreme Value Theorem)**.** *A continuous function on $[a, b]$ has global maximum and global minimum.*

Now we distinguish another type of extrema.

Definition 2.2.3. A point x_0 is a **local maximum** (resp. **local minimum**) of a function f if there is an open interval I of x_0 contained in the domain of f such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in I$. A **local extremum** of f is either a local maximum or a local minimum of f.

A few remarks are in order:

1. Recall a point x_0 is an **interior point** of a set X if there is an open interval I such that $x_0 \in I \subseteq X$.

2. Since an end point of an interval is not in its interior, therefore the local extrema, by definition, cannot happen at the end points.

3. If a global maximum (resp. minimum) happens at an interior point then clearly it must be a local maximum (resp. minimum) as well.

4. Global maximum (minimum) value, if exists, must be unique.

5. If exist, the global maximum value is certainly no less than the global minimum value. But this is not true for local extrema.

Intuitively, local extrema only happen at points with horizontal tangents but as we have seen from examples that they can happen at points where the function is not differentiable. Hence, when looking for extrema we should be focusing on the following type of points.

Definition 2.2.4. A point $c \in \operatorname{dom} f$ is a **critical point** of f if either $f'(c) = 0$ or f is not differential at c.

Note also that any end point in the domain of f must be a critical point of f since the derivative of f does not exist at the end points. The next result, justify our earlier intuition.

Theorem 2.2.5 (Critical Point Theorem)**.** *Local extrema of f only happen at critical points.*

Proof. Without loss of generality, we can assume c is a local minimum (if c is a local maximum of f, then apply the following argument to $-f$). If f is differentiable at c, then for some function φ continuous at c, we have

$$f(x) - f(c) = \varphi(x)(x - c). \tag{2.1}$$

The left-hand side of (2.1) is non-negative on a neighborhood of c, therefore $\varphi(x)$ and $x - c$ have the same sign near c and so $\varphi(c)$ must be 0 by continuity. □

Example 2.2.6. Let f be the restriction of the absolute value on $[-1, 1]$. Then 0 is a global minimum of f and ± 1 are global maxima of f. The global minimum value of f is $f(0) = 0$ and the global maximum value of f is $1 = f(1) = f(-1)$.

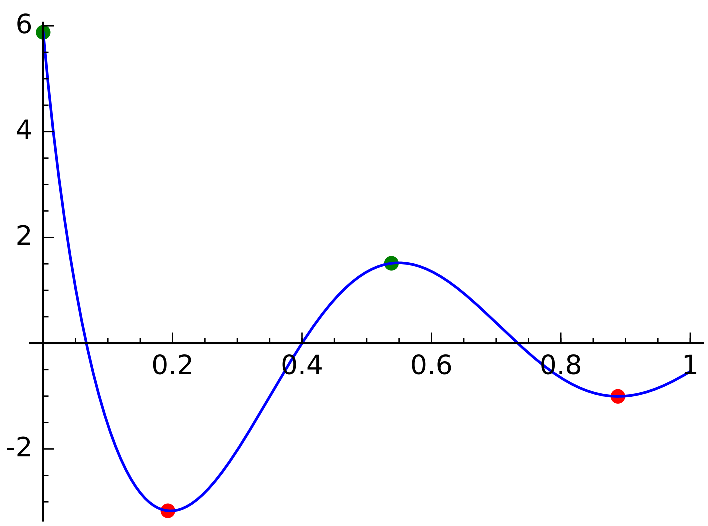


Figure 2.1: The graph of $\cos(3\pi(x + 0.1))/(x + 0.1)$ for $x \in [0, 1]$

According to the critical point theorem, to find the global extrema of a function on $[a, b]$ we need to examine the critical points of f in $[a, b]$, i.e.

1. $c \in [a, b]$ such that f is not differentiable at c. This includes the end-points a, b of the interval.
2. $c \in (a, b)$ such that $f'(c) = 0$.

Note that neither of these conditions implies the critical point is a local extremum as the following examples show.

Example 2.2.7. The function $f(x) = x^3$ has derivative 0 at the origin. However, the origin is clearly not a local extremum.

Example 2.2.8. Consider the function f on $[-1,1]$ defined by $f(x) = x$ for $x \in [-1,0]$ and $f(x) = x/2$ for $x \in [0,1]$. Then f is not differentiable at the origin but again the origin is not a local extremum.

2.3 Concavity

A real-valued function f defined on an interval I is **concave up** (or **convex**) on I if the graph of f is always below any secant with end points over I. In other words, for any $a < b \in I$, $f(c) \leq s(c)$ where s is the secant of f determined by the points $(a, f(a))$ and $(b, f(b))$.

Figure 2.2: The graph of a concave up function

A real-valued function f is **concave down** (or simply **concave**) on I if $-f$ is concave up on I. In other words, f is concave down on I if its graph is always above any secant over I.

Figure 2.3: The graph of a concave down function

The concept of concave does not rely on the concept of differentiability, however, if f is differentiable on I then we can use derivative to capture concavity.

Proposition 2.3.1. *Suppose f is differentiable on I then f is concave up on I if and only if f' is increasing on I.*

Example 2.3.2. It is easy to easy from the graph of $f(x) = x^3$ that f is concave up on $x \geq 0$ and concave down on $x \leq 0$. Note that this is confirmed by Proposition 2.3.1 since $f'(x) = 3x^2$ is increasing on $x \geq 0$ and decreasing on $x \leq 0$.

Proposition 2.3.3. *If f is both differentiable and concave up (or down) on I, then f is continuously differentiable on I.*

Proposition 2.3.4. *If f is concave up (resp. down) and differentiable on I and $f'(c) = 0$ for some $c \in I$, then c is a global minimum (resp. maximum) for f on I.*

Since for differentiable functions, concave up (resp. down) on an interval means f' is increasing, so if f is twice differentiable then the first derivative test implies

Proposition 2.3.5. *A twice differentiable function on an interval I is concave up if and only if $f'' \geq 0$ on I.*

A point $x_0 \in I$ (or rather a point $(x_0, f(x_0)$ on the graph of f) is an **inflection point** of f if the concavity of f on either side of x_0 are different.

2.3.1 Exercises

1. For $x_1, x_2 \in I$, let $m(x_1, x_2)$ be the slope of the secant of the graph of f with end points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Show that f is concave up on I if and only if the function $m(x_1, x_2)$ is increasing in x_1 (resp. x_2) for any fixed x_2 (resp. x_1) in I.

2. Formula a similar equivalent condition for concave down.

Chapter 3

Integration

3.1 Anti-derivatives

So far we have been concerning ourselves with the following question: given a function $f(x)$, how to find its derivative $f'(x)$? In this section, we ask the "inverse" problem, namely given a function $f(x)$ find a function $F(x)$ whose derivative is $f(x)$. We call such a function $F(x)$ an **anti-derivative** of $f(x)$. In other words, $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$. The first thing to note is that, unlike its derivative, a function can have many anti-derivatives. For example, x^2 and $x^2 + 1$ are two anti-derivatives of $2x$. We use the notation

$$\int f(x)\, dx$$

to denote the class of all anti-derivatives of $f(x)$. This class of functions is also called the **indefinite integral** of $f(x)$ with respect to x. The second thing to note is that if $F_1(x)$ and $F_2(x)$ are two anti-derivatives of $f(x)$, then they differ by a function whose derivative vanishes (on the domain of f). This is because $(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$. They are, according to the mean value theorem 2.1.1, precisely the **locally constant functions**. In particular, if the domain of f is connected then these are simply the constant functions (identified with the real numbers). It is customary to denote the class of locally constant functions (with the domain understood) by C. Thus we write

$$\int 2x\, dx = \{x^2 + k \colon k \in \mathbb{R}\} = x^2 + C.$$

Quite often C is also referred to as the **integration constant**. Although, as we have seen, it is the class of locally constant functions.

Example 3.1.1. Let us find the indefinite integral of $1/x$. We know that the derivative of $\ln x$ is $1/x$, so $\int 1/x\, dx$ should be $\ln x + C$. However, is it not completely correct. Care must be paid to the domains of these functions. When we say that $(\ln x)' = 1/x$, it is understood that the domain is $x > 0$ since it is the domain of $\ln x$. However, when we start with $1/x$, it is understood that the domain is $x \neq 0$. So $\ln x$ is only "half" of the anti-derivative of $1/x$ (graph them and you will see). So what is the other half? By symmetry, it is not hard to see that the derivative of $\ln(-x)$ is $1/x$ on $x < 0$. Indeed one can check this by differentiation. Thus $\ln |x|$ is an anti-derivative of $1/x$. Hence

$$\int \frac{1}{x}\, dx = \ln |x| + C.$$

Also, since the domain $x \neq 0$ has two connected pieces hence the class C of locally constant functions on this domain is the class of functions of the form

$$\begin{cases} c_1 & x > 0 \\ c_2 & x < 0 \end{cases}$$

where c_1, c_2 are, not necessarily the same, constants. For example, the function

$$f(x) = \begin{cases} \ln(x) + 2 & x > 0 \\ \ln(-x) - 3 & x < 0 \end{cases}$$

is an anti-derivative of $1/x$.

Exercises

3.1.1 Substitution

Sometime it is not immediate what should be the anti-derivate of a given function. For example, $x \sin(x^2)$. Substitution is an idea that help us to recognize the anti-derivative. To facilitate the subsequent discussion, we introduce **differential forms**. We will only treat them on a formal bases as a tool for integration. Readers who want an in-depth treatment of the topic can consult [give a ref]. We think of differential forms instead of functions as the objects to be integrate. So instead of the function $x \sin(x^2)$, we think of the form $x \sin(x^2)\, dx$ as the entity to be integrated. In general, a differential form (in a single variable x) is an expression of the form $f(x)\, dx$. Given a differentiable function $f(x)$, we write $df(x)$ (or simply df) for the differential form $f'(x)\, dx$. We say that a differential form is **exact** if it is $df(x)$ for some differentiable function $f(x)$.

Integrating exact form is trivial since by definition $f(x)$ is an anti-derivative of $f'(x)$, so we have

$$\int df(x) = \int f'(x)\, dx = f(x) + C$$

Back to our example, to integrate $x\sin(x^2)\, dx$, first we recognize that

$$\sin(u)\, du = d(-\cos(u))$$

is an exact form. (This is just restating the fact that $-\cos(u)$ is an anti-derivative of $\sin(u)$.) It is reasonable to make the substitution $u = x^2$. So $du = dx^2 = 2xdx$ thus

$$x\sin(x^2)\, dx = \frac{1}{2}\sin(u)\, du = -\frac{1}{2}d\cos(u)$$

and we conclude that

$$\int x\sin(x^2)\, dx = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\sin(x^2) + C.$$

Before giving a few more examples on integration by substitution, we would like to point out that to differentiate $\sin(x^2)$, one would substitute x^2 by a variable and applies the chain rule. In fact, integration by substitution can be viewed as the "anti-chain-rule". As just like the chain rule, one gets better in choosing the right substitution for integration with experience.

Exercises

3.2 Definite Integrals

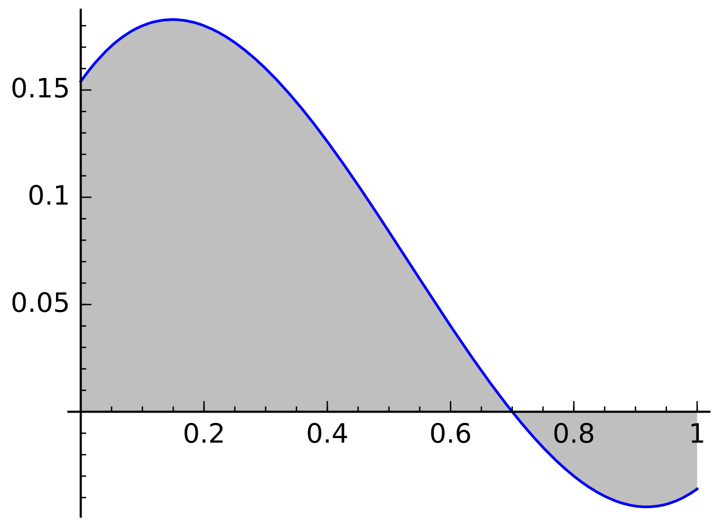
We begin with yet another an imprecise definition: Let $f(x)$ be a function of x defined on an interval $[a, b]$, denoted by

$$\int_a^b f(x)\, dx$$

the *net-area* between the graph of f and the x-axis. We often simply write $\int_a^b f$ if the variable is understood. By net-area here we mean the sum of the areas of the regions above the x-axis minus the sum of the areas of the regions below the x-axis. This definition is vague because it rests on the concept of "area" which is undefined. In fact, once we developed the theory of definite integral carefully, it can be used to give a precise definition of "area". However, our intuition on area will get us some leg in understanding definite integrals. At the moment two things to keep in mind:

1. There are "non-integrable" functions. So roughly speaking that means there are functions to which it does not make sense to talk about the net area between its graph and the x-axis.

2. Continuous functions on a closed bounded interval are integrable.



Moreover, there is a "direction" in the definite integral that is understood: if $a \leq b$, then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{3.1}$$

It follows immediately that for a in the domain of f,

$$\int_a^a f(x)\,dx = -\int_a^a f(x)\,dx \tag{3.2}$$

and hence $\int_a^a f(x)\,dx = 0$. With the net-area interpretation of the definite integral in mind, the following results are intuitive.

Proposition 3.2.1. *If f, g are integrable on $[a, b]$ and $k \in \mathbb{R}$ then $f + g$ and kf are integrable on $[a, b]$. Moreover,*

1. $\int_a^b kf = k\int_a^b f$.

2. $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Proposition 3.2.2. *Suppose f is integrable on $[a,b]$ and $c \in [a,b]$. Then f is integrable on $[a,c]$ and $[c,b]$; moreover*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proposition 3.2.3. *Suppose f is integrable on $[a,b]$ and $f \geq 0$ on $[a,b]$. Then*

$$\int_a^b f \geq 0.$$

Proposition 3.2.4. *If f is integrable on $[a,b]$ and g differs from f at only finitely many points in $[a,b]$, then g is integrable on $[a,b]$; moreover f and g has the same integral over $[a,b]$.*

Exercises

Assume all the functions considered in the following are integrable on $[a,b]$.

1. Show that $\int_a^b (f-g) = \int_a^b f - \int_a^b g$.
2. Show that if $f \leq g$ on $[a,b]$ then $\int_a^b f \leq \int_a^b g$.
3. (**Integral form of the Mean Value Theorem**) Suppose f is continuous on $[a,b]$. Show that there is a $c \in [a,b]$ such that
$$f(c)(a-b) = \int_a^b f.$$
4. (**Translational invariant**) Suppose f is integrable on $[a,b]$ and $k \in \mathbb{R}$. Show that the function $g(x) = f(x-k)$ is integrable on $[a+k, b+k]$. Moreover,
$$\int_{a+k}^{b+k} g(x)\,dx = \int_a^b f(x)\,dx.$$
5. We say that a subset M of $[a,b]$ is **measurable** if its membership function χ_M is integrable on $[a,b]$. The value $\int_a^b \chi_M$ is the **measure** of M. We say that a property holds **almost everywhere** (a.e.) in $[a,b]$ if the subsets that the property fails is a measure zero set.
   (a) Suppose $f = g$ a.e. on $[a,b]$. Show that f is integrable on $[a,b]$ if and only if g is. Moreover, their integrals on $[a,b]$ are equal.
   (b) Show that every singleton subset of $[a,b]$ has measure 0. Deduce that every finite subset of $[a,b]$ has measure 0.
   (c) Deduce Proposition 3.2.4.

Chapter 4

The Fundamental Theorem of Calculus

4.1 The Fundamental Theorems of Calculus

Theorem 4.1.1 (FTC 1st form)**.** *Suppose $f, F\colon [a,b] \to \mathbb{R}$ are functions such that:*

1. *F is continuous on $[a,b]$.*

2. *$F'(x) = f(x)$ for all $x \in [a,b]$.*

Then f is integrable on $[a,b]$ and

$$\int_a^b f = F(b) - f(a)$$

Theorem 4.1.2 (FTC 2nd form)**.** *Suppose f is integrable on $[a,b]$ and F is an antiderivative of f. Then*

1. *F is continuous on $[a,b]$.*

2. *$F'(x) = f(x)$ almost everywhere on $[a,b]$.*

3. *If f is continuous at $c \in [a,b]$, then $F'(c) = f(c)$.*

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