Poisson integral formula

The Poisson integral formula

\[ u(z_0 + re^{i\theta}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(z_0 + Re^{it}) \, dt}{R^2 + r^2 - 2rR\cos(t - \theta)} \] (1)

computes the value of a harmonic function \( u \) at a point \( z = z_0 + re^{i\theta} \) inside the disc of radius \( R \) centered at \( z_0 \) by integrating \( u \) times a weighting function

\[ \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(t - \theta)} \]

called the “Poisson kernel” around the boundary of the disc. (See Caine’s textbook p. 7.7). The proof in Caine seems to come out of nowhere so I thought it might be useful to explain how an ordinary mortal could produce this formula. Of course Siméon Denis Poisson (1781-1840) was no ordinary mortal; you may read about him in Wikipedia.

The basic idea of the Poisson integral formula comes from an averaging formula called the **Mean-value property of harmonic functions** that says the value of the harmonic function \( u \) at the center \( z_0 \) of a disc of radius \( R \) is the average of its values around the boundary of the disc.

\[ u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{it}) \, dt \] (2)

One consequence of the mean-value property is that if \( m \) is the minimum value of \( u \) on the boundary of the disc, and \( M \) is its maximum value on the boundary of the disc, then its value at the center must lie between \( m \) and \( M \). This is true because the average of any set of real numbers lies between the minimum and the maximum of the numbers. And, since the mean-value property is true for every disc, no matter what \( R \) is, it follows that a harmonic function cannot have a local minimum, or a local maximum, in the interior of its domain unless the function is constant.

Why is the mean-value property true?

It is a consequence of the Cauchy integral formula. (What else could it be?) To see this recall that on a simply connected region (nonempty, open, with no holes) every harmonic function is the real part of a holomorphic function. So on a region containing our disc we may assume \( u(z) \) is the real part of some holomorphic function

\[ f(z) = u(z) + iv(z) \]

where \( v \) is another real-valued harmonic function. The Cauchy integral formula says

\[ u(z_0) + iv(z_0) = f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z) \, dz}{z - z_0} = \frac{1}{2\pi i} \int_C \frac{(u(z) + iv(z)) \, dz}{z - z_0} = \frac{1}{2\pi i} \int_C \frac{u(z) \, dz}{z - z_0} + \frac{1}{2\pi i} \int_C \frac{iv(z) \, dz}{z - z_0} \] (3)
if $z_0$ is inside the simple closed curve $C$. Let $C$ be the boundary of the disc of radius $R$ centered at $z_0$. Parametrize $C$ by

$$z(t) = z_0 + Re^{it}, \text{ for } 0 \leq t \leq 2\pi$$

so

$$z(t) - z_0 = Re^{it}$$

$$dz = iRe^{it} \, dt.$$ 

Substitute these into equations (3).

$$u(z_0) + iv(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} u(z(t)) \frac{iRe^{it}dt}{Re^{it}} + \frac{1}{2\pi i} \int_0^{2\pi} iv(z(t)) \frac{iRe^{it}dt}{Re^{it}}$$

The $Re^{it}$-terms cancel and the $i$ in $2\pi i$ cancels the $i$ in $i \, dt$, so

$$u(z_0) + iv(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z(t)) \, dt + \frac{i}{2\pi} \int_0^{2\pi} v(z(t)) \, dt$$

The left integral is real because $u$ is real, and the right integral is imaginary because $v$ is real. Thus one can separate real and imaginary parts:

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z(t)) \, dt$$

$$v(z_0) = \frac{1}{2\pi} \int_0^{2\pi} v(z(t)) \, dt.$$ 

This proves the mean value property, equation (2), for the harmonic function $u$ (and also for $v$).

The mean value property is a special case of the Poisson integral formula (1) where $u$ is evaluated at the center of the disc. (To see this set $r = 0$ in equation (1).)

To derive the full Poisson integral formula one must move away from the center of the disc. The easiest way to do that is to use a one-to-one holomorphic function $g$ that maps the disc to itself, but moves the center $z_0$ of the disk to another point inside the disk.

In the figure below $g$ maps the unit disc centered at 0 to itself, pushing points inside the disc toward the right side so $g(0) = 1/2$. 

![Diagram](image-url)
Now $u(z)$ is a harmonic function of $z$, so it has a “harmonic conjugate” function $v(z)$ so the sum $u(z) + iv(z)$ is an analytic function of $z$. The composition $u(g(z)) + iv(g(z))$ is also analytic because the chain rule says the composition of two differentiable is also differentiable, and this also works for complex differentiation. Therefore the real part $u(g(z))$ is a harmonic function of $z$.

Let $w = g(z_0)$. Since $u(g(z))$ is a harmonic function of $z$ he mean value property says

$$u(w) = u(g(z_0)) = \frac{1}{2\pi} \int_0^{2\pi} u(g(z(t))) \, dt$$

where

$$z(t) = z_0 + Re^{it}$$

as before.

This is almost the Poisson integral formula, but not quite, because the Poisson integral uses $u(z(t))$ not $u(g(z(t)))$. So we have to figure out a way to get rid of the annoying $g$ on the right hand side.

We’ll use integration-by-substitution to eliminate $g$. Assume that $g$ maps the boundary circle

$$z(t) = z_0 + Re^{it}, \quad 0 \leq t \leq 2\pi$$

to itself, perhaps moving points around on the circle, so for each angle $t$ that we can write

$$g(z_0 + Re^{it}) = z_0 + Re^{is}$$

for some angle $s$, where $0 \leq s, t \leq 2\pi$. Then

$$z_0 + Re^{it} = g^{-1}(z_0 + Re^{is}).$$

At this point it will simplify the notation (a lot!) if we call the inverse function $h$ so

$$h = g^{-1} \quad \text{and} \quad z_0 + Re^{it} = h(z_0 + Re^{is}).$$

Differentiating, one has

$$iRe^{it} \, dt = h'(z_0 + Re^{is})iRe^{is} \, ds$$

But $z_0 + Re^{it} = h(z_0 + Re^{is})$ so $Re^{it} = h(z_0 + Re^{is}) - z_0$. Plug that into equation (8) to obtain

$$i \left( h(z_0 + Re^{is}) - z_0 \right) \, dt = h'(z_0 + Re^{is})iRe^{is} \, ds.$$

Divide by $i$ then solve for $dt$

$$dt = \frac{h'(z_0 + Re^{is})}{h(z_0 + Re^{is}) - z_0} \, Re^{is} \, ds. \quad (9)$$
We can now perform the integration by-substitution. Equations (5) and (6) say that
\[ g(z(t)) = g(z_0 + Re^{it}) = z_0 + Re^{is} \]
so, using equation (9) and the fact that \( s \) runs from 0 to \( 2\pi \), the integral formula (4) becomes
\[ u(w) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{is}) \left( \frac{h'(z_0 + Re^{is})}{h(z_0 + Re^{is}) - z_0} \right) Re^{is} \, ds. \]  

(10)

The main thing that’s left is to find a formula for \( h \). We’ll use a linear fractional transformation.

Recall that
1. Linear fractional transformations are analytic and they have inverses, which are also linear fractional transformations.
2. A linear fractional transformation is completely determined by its values at three points. In other words, given two sets of three distinct \( P_1, P_2, P_3 \) and \( Q_1, Q_2, Q_3 \) (all in the complex plane) there is exactly one linear fractional transformation \( h \) such that \( h(P_1) = Q_1, h(P_2) = Q_2 \) and \( h(P_3) = Q_3 \).
3. Linear fractional transformations map circles and lines to circles and lines.
4. Circles and lines are also uniquely determined by three points: given three points in the plane there is exactly one circle or line that contains all three points.

We’re looking for a linear fractional transformation \( h \) that maps the circle \( C \) of radius \( R \) and center \( z_0 \) to itself, and maps the center \( z_0 \) to another given point \( w \) inside the circle.

If \( w = z_0 \) then the identity function \( h(z) = z \) does this, so assume \( w \neq z_0 \).

Let \( L \) be the line through \( z_0 \) and \( w \), and let \( A \) and \( B \) be the points where \( L \) intersects \( C \).

Let \( h \) be the linear fractional transformation such that
\[ h(w) = z_0 \text{ and } h(A) = A \text{ and } h(B) = B. \]
is the unique circle or line that contains the three points \( w, A \) and \( B \). Thus \( h(L) \) is the unique circle or line that contains \( h(w) = z_0, f(A) = A, \) and \( f(B) = B \). But \( L \) contains these three points so we must conclude that 
\[
h(L) = L.
\]
(This means that \( h \) may move points in \( L \) to other places inside \( L \) but it can’t move them to places outside \( L \).)

\( C \) is the unique circle or line that intersects \( L \) at 90° angles at the points \( A \) and \( B \). \( h \) is a conformal map (it preserves angles) so \( h(C) \) is the unique circle or line that meets the line \( h(L) = L \) at 90° angles at the points \( h(A) = A \) and \( h(B) = B \). Thus 
\[
h(C) = C.
\]

This \( h \) is the right function for our purposes. To find a formula for it, start by finding formulas for linear fractional transformations \( h_1 \) and \( h_2 \) such that 
\[
1 = h_1(w) = h_2(z_0), \quad 0 = h_1(A) = h_2(A) \quad \text{and} \quad \infty = h_1(B) = h_2(B). \quad \text{Then} \quad h = h_2^{-1} \circ h_1 \quad \text{is the function we want. The algebra isn’t hard, you may work it out yourself using the trick for writing down formulas for} \ h_1 \ \text{and} \ h_2 \ \text{that we discussed in class. I’ll just give the result. To put the result in the most convenient form let} \ \theta \ \text{be the angle between} \ L \ \text{and a horizontal line, and let} \ r = |w - z_0| \ \text{and} \ R = |A - z_0| \ \text{(see the figure above). Then} \ w = z_0 + re^{i\theta} \ \text{and} \ A = z_0 + Re^{i\theta}. \]

and 
\[
h(z) = z_0 + R^2 \frac{re^{i\theta} - (z - z_0)}{r(z - z_0)e^{-i\theta} - R^2}. \quad (11)
\]
Computing the quotient \( h'(z)/(h(z) - z_0) \) in equations (9) and (10) looks like a mess, but here we get a lucky break because \( z_0 \) is constant so 
\[
\frac{h'(z)}{h(z) - z_0} = \frac{d}{dz}(h(z) - z_0) = \frac{d}{dz} \log (h(z) - z_0).
\]
(the last line comes from the chain rule. Remember “logarithmic differentiation” from calculus I?). This enables us to use properties of logarithms to simplify the calculations.
\[
\log(h(z) - z_0) = \log \left( R^2 \frac{re^{i\theta} - (z - z_0)}{r(z - z_0)e^{-i\theta} - R^2} \right) = \log(R^2) + \log \left( re^{i\theta} - (z - z_0) \right) - \log \left( r(z - z_0)e^{-i\theta} - R^2 \right).
\]

\( R^2 \) is constant so 
\[
\frac{h'(z)}{h(z) - z_0} = \frac{d}{dz} \log (h(z) - z_0) = \left( -\frac{1}{re^{i\theta} - (z - z_0)} - \frac{re^{-i\theta}}{r(z - z_0)e^{-i\theta} - R^2} \right)
\]
Substitute $z = z_0 + R e^{i\theta}$ into the last equation to obtain

$$
\frac{h'(z_0 + R e^{i\theta})}{h(z_0 + R e^{i\theta}) - z_0} = \frac{-1}{r e^{i\theta} - R e^{i\theta}} - \frac{r e^{-i\theta}}{R e^{i\theta} e^{-i\theta} - R^2}
$$

$$
= \frac{-1}{r e^{i\theta} - R e^{i\theta}} - \frac{r e^{-i\theta}}{R^2 - r R e^{i(s-\theta)} - r e^{-i\theta} (r e^{i\theta} - R e^{i\theta})}
$$

$$
= \frac{r^2 R e^{i\theta} - r R e^{i(s-\theta)} - r^2 + r R e^{i(s-\theta)}}{R^2 - r^2}
$$

$$
= \frac{R^2 - r^2}{r^2 R e^{i\theta} - r R e^{i(s-\theta)} - r^2 + r R e^{i(s-\theta)}}
$$

Multiply by $R e^{i\theta}$ to obtain the expression $\left( \frac{h'(z_0 + R e^{i\theta})}{h(z_0 + R e^{i\theta}) - z_0} \right) R e^{i\theta}$ that occurs in the integrand of formula (10):

$$
\left( \frac{h'(z_0 + R e^{i\theta})}{h(z_0 + R e^{i\theta}) - z_0} \right) R e^{i\theta} = \frac{(R^2 - r^2) R e^{i\theta}}{r^2 R e^{i\theta} - r R e^{i(s-\theta)} - r^2 e^{-i\theta} + R^3 e^{i\theta}}
$$

$$
= \frac{(R^2 - r^2)}{r^2 - r R e^{i(s-\theta)} + R e^{i(s-\theta)} + R^2 e^{i\theta}}
$$

But

$$
e^{i(s-\theta)} + e^{i(\theta-s)} = \cos(s - \theta) + i \sin(s - \theta) + \cos(\theta - s) + i \sin(\theta - s)
$$

$$
= \cos(s - \theta) + i \sin(s - \theta) + \cos(-s + \theta) + i \sin(-s + \theta)
$$

$$
= 2 \cos(s - \theta)
$$

because $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$ for every $z$. Thus

$$
\left( \frac{h'(z_0 + R e^{i\theta})}{h(z_0 + R e^{i\theta}) - z_0} \right) R e^{i\theta} = \frac{(R^2 - r^2)}{r^2 - 2 r R \cos(s - \theta) + R^2 e^{i\theta}}
$$

Substituting this into the integral formula (10) we obtain the Poisson integral formula.