Rank Inequalities in the Theory of Differentially Closed Fields

Wai Yan Pong
California State University Dominguez Hills

May 7, 2004

Let $m \geq 1$ be an integer. The theory of differentially closed fields with m commuting derivations (m-DCF_0) has been actively studied recently [11, 14, 2, 1] and [12]. Readers who are interested in the model theory of differential fields and its applications can consult [8, 9, 13] and [21]. In this article, I limit myself to give a "road-map" of the proof of the following result:

Theorem. *Let p be a complete n-type in m-DCF_0 over a differential field K. Suppose d and e are the typical differential dimension and the differential type of p respectively and that $e \geq 1$. Then $1 \leq d \leq n$ and*

$$\omega^e d \leq \mathrm{RU}(p) \leq \mathrm{RM}(p) \leq \mathrm{RH}(p) \leq \mathrm{RD}(p) < \omega^e(d+1). \qquad (*)$$

For 1-types over an ordinary differential field K, the inequalities are due to Poizat [15]. Their generalization to n-types are fairly easy and were obtained by the author in [17] with RH replaced by Δ-dimension and $\mathrm{RD}(p)$ replaced by $\omega d + b$ where $dT + b$ is the Kolchin polynomial of the type p. Since both d and b in that case are natural numbers so the map $dT + b \mapsto \omega d + b$ induces a well-order on the set of Kolchin polynomials. At that time, however, I did not realize that this assignment can be regarded as a generalization of RD. Later Benoist generalized the definitions of both RH and RD to n-types over ordinary differential fields and proved the inequalities for these ranks in [3]. His definition of the rank RD is different from ours, we will address this issue in Section 2. Let us also note that each of the inequalities appeared can be strict (even in the ordinary case). Examples that illustrate these phenomena are nontrivial except for the first inequality (see § 3). In particular, the question of whether RU=RM in DCF_0 was open for quite a while until it was settled by Hrushovski and Scanlon in [5].

The proof of these inequalities become considerably harder when the differential field is equipped with more than one derivation. McGrail in her thesis (see [11]) succeeded in proving the theorem without RD and with RH replaced by the Δ-dimension. So our primary goal here is to clarify the relationship between the rank RD, RH and Δ-dimension. The use of Kolchin polynomials in McGrail works is prominent. In fact, with a perfect hindsight one sees that she had already developed the results of RD that are necessary in proving these inequalities.

One of the difficulties in generalizing the definition of RD to the several derivations case is that the coefficients of the Kolchin polynomials in this case are merely rational numbers. Therefore, it is no longer obvious that the simple idea of assigning the Kolchin polynomials to their values at ω will yield a well ordering on them. It was not until a meeting in 2000 that I first learned that the Kolchin polynomials are well ordered by dominance. Again with a perfect hindsight, this result of Sit [20] can be seen as an easily consequence of the results in [11] (Lemma 4.2.13 and Proposition 4.2.15). At the same meeting, Scanlon suggested that one should investigate the "meanings" of the coefficients of the Kolchin polynomials. Around the same time, an application of McGrail's version of the theorem also prompted me to take a closer look of it again: it follows immediately from these inequalities that RU = RM for types with RM of the form $\omega^e d$ with $e \geq 1$. This fact is a crucial ingredient in showing that RU=RM for generic types of definable groups in m-DCF_0 [14]. On the other hand, Benoist in [3] showed that RM and RH of a definable group in DCF_0 can be different. Moreover, he showed that the notion of RM-generic and RH-generic are different.

In the first version of [2], I defined the rank RD using sequence of minimizing coefficients (see § 2) and proved a result about the RD of the fibers in definable families. The proofs there used some algebra of monomial ideals. However, it is Aschenbrenner who first recognized the full potential of relating the study of the Kolchin polynomials to that of the monomial ideals. Later through our study of Noetherian orderings we are able to give a conceptual proof of Sit's Theorem and a definition of RD that works in the several derivations case [1, 2].

Acknowledgement. I thank the referee for many useful suggestions and comments.

1 Preliminaries

We start with a brief review of the facts about differential algebra and the model theory of differential fields that will be used later. Readers may find it helpful to have [6, 7] and [11] at hand.

By a *differential ring* we mean a commutative ring with 1 equipped with m commuting derivations. We use $\Delta = \{\delta_1, \ldots, \delta_m\}$ to denote the set of derivations. A *Ritt ring* is a differential ring containing the field of rational numbers. A *differential field* is a Ritt ring which is also a field. In particular, it has characteristic 0. We use the adjective "ordinary" to emphasize the case $m = 1$. Let R be a differential ring, an ideal I of R is a *differential ideal* if it is closed under the derivations, i.e. $\delta_i I \subseteq I$ for all $1 \leq i \leq m$. An ideal I is *perfect* if it equals to its own radical. In a Ritt ring, the radical of a differential ideal is still a differential ideal [7, Theorem 3.2.12]. Moreover, any maximal differential ideal of a Ritt ring is prime and the nilradical of a Ritt ring is the intersection of all prime differential ideals [7, Corollary 3.2.20]. In particular, every Ritt ring possesses a differential prime ideal. We use $\operatorname{Spec} R$ and $\Delta\text{-}\operatorname{Spec} R$ to denote the set of prime ideals and the set of differential prime ideals of R respectively.

Let Θ be the free commutative monoid generated by the derivations. A typical element of Θ is of the form $\delta_1^{e_1} \cdots \delta_m^{e_m}$ where the e_i's are non-negative integers. The sum of the e_i's is called the *order* of $\delta_1^{e_1} \cdots \delta_m^{e_m}$. Let a be an element of a differential ring R. A *derivative* of a is an element of R of the form θa for some $\theta \in \Theta$. The *order* of a derivative b of a is defined to be the smallest integer s such that $b = \theta a$ for some $\theta \in \Theta$ of order s. Let L be a differential field extension of a differential field K. A subset A of L is *differentially independent* over K if the set of derivatives of elements of A is algebraically independent over K. In the case where A is a singleton $\{a\}$, we say that the element a is *differentially transcendental* over K. An element is *differentially algebraic* over K if is not differentially transcendental over K.

Let R be a differential ring. The *differential polynomial ring over R with variables $y_1, \ldots, y_n$* is the polynomial ring

$$R\{y_1, \ldots, y_n\} := R[\theta y_j \colon \theta \in \Theta, 1 \leq j \leq n].$$

The derivations $\delta_1, \ldots, \delta_m$ on R extend naturally to $R\{\bar{y}\} = R\{y_1, \ldots, y_n\}$ making it into a differential ring extension of R. In general differential polynomial rings, unlike their algebraic counterparts, are not Noetherian. However, it follows from the Ritt-Raudenbush basis theorem (or the differential basis theorem) [7, Theorem 3.2.23, 5.3.17] that the set of perfect differential ideals of a differential polynomial ring over a differential field satisfies the ascending chain condition.

Let $\mathfrak{L}_m$ be the language of fields $\{+, -, \cdot, ^{-1}, 0, 1\}$ together with the set of unary function symbols $\Delta = \{\delta_1, \dots, \delta_m\}$. From the model theoretic point of view, a differential field can be regarded as an $\mathfrak{L}_m$ structure with the symbols interpreted as the usual field operations and the δ_i's as derivations. A differential field is *differentially closed* if it is existentially closed in the sense of model theory. It turns out that the class of differentially closed fields is elementary. We use m-DCF_0 to denote the common first order theory of differentially closed fields in $\mathfrak{L}_m$. A relatively simple set of axioms for m-DCF_0 can be found in [11, §3]. Recently, a "coordinate free" approach to the axiomatization of the class of differentially closed fields is given by Pierce [12].

The theory m-DCF_0 admits quantifiers elimination [11, Theorem 3.1.7]. A consequence of this fact is the "type-ideal correspondence". Let $p \in S_n(K)$ be a complete n-type over a differential field K. Denote by I_p the set

$$\{f \in K\{y_1, \dots, y_n\} : \text{“}f(y_1, \dots, y_n) = 0\text{”} \in p\}.$$

One checks directly that I_p is a prime differential ideal. It follows from the quantifier elimination of m-DCF_0 that the association $p \leftrightarrow I_p$ is a 1-1 correspondence between $S_n(K)$ and $\Delta\text{-}\mathrm{Spec}\, K\{y_1, \dots, y_n\}$. Moreover, one can verify readily that a tuple is a realization of a type $p \in S(K)$ if and only if its vanishing ideal over K is I_p. Using the type-ideal correspondence and the differential basis theorem, one can show that m-DCF_0 is ω-stable by a type-counting argument [11, Theorem 3.2.1].

It is convenient to fix a universal domain U of m-DCF_0 and consider all differential fields other than U as its small subfields. For each $n \in \mathbb{N}$, we equip U^n with the *Kolchin topology.* The closed sets in this topology are of the form

$$V(S) = \{a \in U^n \colon f(a) = 0 \text{ for all } f \in S\}$$

where S is a subset of $U\{y_1, \dots, y_n\}$. It is easy to check that $V(S) = V(\{S\})$ where $\{S\}$ is the prefect differential ideal in $U\{y_1, \dots, y_n\}$ generated by S. On the other hand, given X a Kolchin closed subset of U^n, the set

$$I(X) = \{f \in U\{y_1, \dots, y_n\} \colon f(a) = 0, \text{ for all } a \in X\}$$

is a perfect differential ideal of $U\{y_1, \dots, y_n\}$. A Kolchin closed set is *irreducible* if it not empty and is not a union of two proper nonempty closed subsets. Let X be a Kolchin closed set, a maximal irreducible Kolchin closed subset of X is called an *irreducible component* of X. Let K be a differential field, a Kolchin closed subset of U^n is *K-closed* if it is of the form $V(S)$ for some $S \subseteq K\{y_1, \dots, y_n\}$. It is not hard to see that the K-closed sets form a topology on U^n. We understand the terms *K-irreducible* and *K-irreducible component* in the obvious way. Let L be a differential field extension of K, then the K-closed subsets of U^n induces a topology on L^n. We write $V_L(S)$ for the set $L^n \cap V(S)$ and $I_L(V)$ for the perfect differential ideal $I(V) \cap L\{y_1, \dots, y_n\}$ of $L\{y_1, \dots, y_n\}$. In the case when L is differentially closed, the differential nullstellensatz [6, Corollary 1, Theorem 2, p.148] states that $X \mapsto I_L(X)$ is an inclusion reserving 1-1 correspondence between the L-closed subsets of L^n and the perfect differential ideals of $L\{y_1, \dots, y_n\}$.

2 RH, RD and Δ-dimension

All results in this section, besides the definition of RD and Proposition 2.10, are either folklore or appear in some form in [4, 11].

In Section 1, we have remarked that the differential spectrum of a differential polynomial ring over a differential field satisfies the ascending chain condition. In general, we can make the following definition:

Definition 2.1. Let R be a differential ring. Suppose $\Delta\text{-Spec}\, R$ satisfies the ascending chain condition then for $P \in \Delta\text{-Spec}\, R$, we define inductively the *Δ-dimension of* P, denoted by $\Delta\text{-dim}\, P$, to be the ordinal

$$\sup\{\Delta\text{-dim}\, Q + 1 : Q \in \Delta\text{-Spec}\, R, Q \supsetneq P\}.$$

In particular, maximal elements of $\Delta\text{-Spec}\, R$ have Δ-dimension 0. For $p \in S_n(K)$, we define its Δ-dimension, $\Delta\text{-dim}\, p$, to be

$$\sup\{\Delta\text{-dim}\, P : L \supseteq K, P \in \Delta\text{-Spec}\, L\{\bar{y}\}, P \cap K\{\bar{y}\} = I_p\}.$$

Poizat introduced the differential height (RH) for 1-types over ordinary differential fields in [15]. Benoist generalized this definition to n-types over ordinary differential fields in [4] (see also [3]). His definition works equally well in the several derivations case without any modification.

Definition 2.2. Let $p \in S(K)$, the ordinal $\mathrm{RH}(p)$ is defined inductively as follows:

- For α a limit ordinal, $\mathrm{RH}(p) \geq \alpha$ if $\mathrm{RH}(p) \geq \beta$ for all $\beta < \alpha$.
- $\mathrm{RH}(p) \geq \alpha + 1$ if there exist L a differential field extension of K and $q, r \in S(L)$ such that q is an extension of p to L, $I_q \subsetneq I_r$ and $\mathrm{RH}(r) \geq \alpha$.
- $\mathrm{RH}(p) = \alpha$ if $\mathrm{RH}(p) \geq \alpha$ and $\mathrm{RH}(p) \not\geq \alpha + 1$.

The following proposition records some basic properties of RH and Δ-dimension. The first two of them are satisfied by every notion of rank [16, Chapter 17].

Proposition 2.3. *Let $L \supseteq K$ be differential fields.*

1. *Both RH and Δ-dimension are invariant under automorphisms of differential fields.*
2. *Let $p \in S(K)$ and q be an extension of p over L, then* $\mathrm{RH}(p) \geq \mathrm{RH}(q)$.
3. *Suppose $p_1, p_2 \in S(K)$ and $I_{p_1} \subsetneq I_{p_2}$ then* $\mathrm{RH}(p_1) > \mathrm{RH}(p_2)$.
4. $\Delta\text{-}\dim I_p \leq \mathrm{RH}(p)$.

Proof. Both (1) and (2) follow immediately from the definitions. For (3), one simply takes $L = K$, $q = p_1$ and $r = p_2$ in the definition of RH. To show (4), suppose $\Delta\text{-}\dim I_p \geq \alpha$; then for each $\beta < \alpha$ there exists $I_q \supsetneq I_p$ such that $\Delta\text{-}\dim I_q \geq \beta$. By induction hypothesis, $\mathrm{RH}(q) \geq \beta$. Hence by (3), $\mathrm{RH}(p) \geq \beta + 1$. Since $\beta < \alpha$ is arbitrary, we conclude that $\mathrm{RH}(p) \geq \alpha$. □

The differential order (RD) was also introduced by Poizat in for 1-types over ordinary differential fields in [15] (see also [16, Chapter 6]). In that case, $\mathrm{RD}(a/K)$ is defined to be the order of the minimal polynomial of the vanishing ideal of a over K. When a is differentially algebraic over K, $\mathrm{RD}(a/K)$ coincides with the transcendence degree over K of the differential field generated by a and K. Recently, a generalization of RD to n-types in m-DCF_0 has been worked out by Aschenbrenner and the author in [2]. Here we will only give a quick introduction to this rank and refer our readers to [1] and [2] for a thorough treatment.

A polynomial $f(T) \in \mathbb{Q}[T]$ is *numerical* if $f(s)$ is an integer for all sufficiently large integer s. For $f(T), g(T) \in \mathbb{Q}[T]$, we say that f *dominates* g, denoted by $f \geq g$, if $f(s) \geq g(s)$ for all sufficient large integer s. It is straight forward to check that dominance is a total order on $\mathbb{Q}[T]$.

Let a be a tuple in some differential extension of K. Denote by $K\langle a\rangle_s$ the field generated over K by the derivatives of a of order at most s. For sufficiently large s, the transcendence degree of $K\langle a\rangle_s$ over K is given by a numerical polynomial [6, Theorem 6, p.115]. We call this polynomial the *Kolchin polynomial of a over K* and denote it by $\chi_{a/K}$. It is easy to see that $\chi_{a/K}$ is completely determined by the type of a over K; so we can define, χ_p, the *Kolchin polynomial of a type $p \in S(K)$* to be $\chi_{a/K}$ where a is any realization of p. Using the type-ideal correspondence (see § 1), we define the *Kolchin polynomial of a prime differential ideal* to be the Kolchin polynomial of its corresponding type. Let p be an n-type over K. Since χ_p is a numerical polynomial, it can be written as $\sum_{i=0}^{e} a_i \binom{T+i}{i}$ where $a_i \in \mathbb{Z}$ (e.g. see [19, Lemma 1 Ch.II B]). We call e, the degree of the polynomial χ_p, the *differential type of p* and a_e the *typical differential dimension of p*.[1] By Theorem 6 of [6, p.115], $e \leq m$ the number of derivations and $0 \leq a_e \leq n$. Moreover, $a_e = 0$ if and only if p is an algebraic type. By a theorem of Sit [20, Proposition 5] (also see [1]), the set of Kolchin polynomials is well-ordered by dominance. The order type of the set of Kolchin polynomials is ω^{m+1} [1, §3].

Definition 2.4. The differential order of a tuple a over K, denoted by $\mathrm{RD}(a/K)$ is defined to be the ordinal corresponding to $\chi_{a/K}$ under the dominance order. Similarly, we define the *differential order of a type* (*a differential prime ideal*) to be the ordinal corresponds to its Kolchin polynomial under the dominance order.

For each Kolchin polynomial χ, there is a tuple $(b_e, \ldots, b_0) \in \mathbb{N}^e$ where e is the degree of χ such that $\mathrm{RD}(\chi) = \sum_{i=0}^{e} \omega^i b_i$ [1, § 3]. The tuple $(b_e, \ldots, b_0)$ is called the *sequence of minimizing coefficients* of χ (see Definition 2.4.9 and Proposition 2.4.10 in [7]). As an example, let us compute the differential order of a linear Kolchin polynomial $\chi(T) = dT + b$. Write $\chi(T)$ as $d\binom{T+1}{1} + (b-d)$. Then according to [7, Definition 2.4.9], the sequence of minimizing coefficients of χ is (d, v) where

$$v = \chi(T+d) - \binom{T+d+2}{2} + \binom{T+2}{2} = \binom{d}{2} + (b-d).$$

[1] The differential type and the typical differential dimension of a type over K are called the K-type and K-degree in [11]. However, the latter terminology may cause confusion when used in conjunction with various model-theoretic notions.

So $\mathrm{RD}(\chi) = \omega d + \binom{d}{2} + (b-d)$. In particular, if a is differentially transcendental over an ordinary differential field K, then $\chi_{a/K} = T+1$ and hence $\mathrm{RD}(a/K) = \omega$.

Let us explain the relationship between our definition of RD and the one given by Benoist in [3]. As we have seen, the RD of $dT+b$ is $\omega d + \binom{b}{2} + (b-d)$ while according to Benoist's definition it will be $\omega d + (b-d)$. The discrepancy here is due to the fact that the Kolchin polynomials considered in [3] only come from types over ordinary differential fields. They form a proper subset of the set of all linear Kolchin polynomials, for example in the ordinary case the constant term of a Kolchin polynomial is always non-negative, in fact it never smaller than the leading coefficient.

Let us also point out that Kolchin polynomial is not a differential birational invariant, i.e. two tuples may generate the same differential field over K yet their Kolchin polynomials over K are different. This can be seen from the following "silly" example: Let a be differentially transcendental over an ordinary differential field K. Clearly a and the pair (a, a') are differential bi-rational with each other. But their Kolchin polynomials over K are $T+1$ and $T+2$ respectively. This phenomenon can also arise from the interplay between the derivations. An example can be found in [7, Example 2.4.5].

Our next proposition gathers some basic properties of RD.

Proposition 2.5. *Let p and q be complete types over K and L respectively.*

1. *q extends p if and only if I_q lies over I_p.*
2. *If q extends p then $\chi_p \geq \chi_q$ hence $\mathrm{RD}(p) \geq \mathrm{RD}(q)$.*
3. *Let $P \in \Delta\text{-}\mathrm{Spec}\, K\{\bar{y}\}$. There are finitely many minimal prime differential ideals in $L\{\bar{y}\}$ containing the perfect ideal generated by P in $L\{\bar{y}\}$. If Q is one of them then Q lies over P and $\chi_P = \chi_Q$.*

Proof. Statement (1) follows immediately from quantifier elimination. Statement (3) is the characteristic 0 case of [6, Proposition 3(b), p.131]. Finally, if q is an extension of p, by (1) I_q lies over I_p. It follows from (3) and the type-ideal correspondence that there exists a type p' over L such that $I_{p'} \subseteq I_q$ is lying over I_p. Moreover $\chi_{p'} = \chi_p$. So by [6, Proposition 2, p.130], $\chi_q \leq \chi_{p'} = \chi_p$. □

A prime differential ideal Q satisfying (3) in the above proposition is called *a prime differential component* of P over L or simply *an L-component* of P.

Lemma 2.6. *Let L be a $|K|^+$-saturated differentially closed field containing K. Suppose V is a K-irreducible closed set. Then the L-irreducible components of V are conjugate with each other under $\mathrm{Aut}_K(L)$, the differential automorphisms group of L over K.*

Proof. Since V is defined over K, $\mathrm{Aut}_K(L)$ acts on the L-irreducible components of V. There are only finitely many L-irreducible components of V, therefore the union of the elements of each orbit is an L-closed set. These unions are stabilized by $\mathrm{Aut}_K(L)$ and since L is $|K|^+$-saturated, we conclude that they are all K-closed. The union of all these K-closed sets is V itself and since V is K-irreducible therefore $\mathrm{Aut}_K(L)$ must act transitively on the L-irreducible components of V. □

Using the differential nullstellensatz, we obtain the following algebraic version of Lemma 2.6: Let L/K be the same as stated in the lemma. Let P be a prime differential ideal of $K\{y_1, \ldots, y_n\}$, then $\mathrm{Aut}_K(L)$ acts transitively on the L-components of P. One can also prove this directly by using the several derivations version of Corollary 3.6 in [8].

We say that the *Going-Down theorem for (differential) prime ideals* holds for a (differential) ring extension $A \subseteq B$ if for $P_2 \subsetneq P_1$ (differential) prime ideals of A and Q_1 a (differential) prime ideal of B lying over P_1, there exists Q_2 a (differential) prime ideal of B contained in Q_1 lying over P_2.

Proposition 2.7. *Let L/K be a differential field extension. The Going-Down theorem for differential prime ideals holds for the differential ring extension $K\{y_1, \ldots, y_n\} \subseteq L\{y_1, \ldots, y_n\}$.*

Proof. Since the natural inclusion $K \hookrightarrow L$ is a flat map and flatness is stable under base change, therefore $B := L\{y_1, \ldots, y_n\} = L \otimes_K K\{y_1, \ldots, y_n\}$ is flat over $A := K\{y_1, \ldots, y_n\}$. So the Going-Down Theorem for prime ideal holds for this extension [10, Theorem 9.5]. Therefore, if $P_2 \subsetneq P_1$ are two prime differential ideals in A and Q_1 is a prime differential ideal in B lying over P_1, then the differential ring $B_{Q_1} \otimes_{A_{P_1}} \kappa(P_2 A_{P_1})$ has a prime ideal and hence it is not the zero ring. Thus the canonical map from Q into this ring is injective so it is a Ritt ring and therefore possesses a prime differential ideal (see § 1). The preimage of this prime differential ideal in B is contained in Q_1 and lying over P_2. □

Our study of RH and RD leads to the following characterization of forking in m-DCF_0 (see also [11, Theorem 4.3.10]).

Proposition 2.8 (Characterization of Forking). *Let $K \subseteq L$ be an extension of differential fields. Let $p \in S(K)$ and $q \in S(L)$ be an extension of p. Then the following are equivalent:*

1. *q is a non-forking extension of p.*

2. *$\chi_q = \chi_p$.*

3. *$\mathrm{RD}(p) = \mathrm{RD}(q)$.*

4. *I_q is an L-component of I_p.*

5. *$\mathrm{RH}(p) = \mathrm{RH}(q)$.*

Proof. (1) $\iff$ (2) follows from (2) of Proposition 2.5 and the equivalence of (1) and (3) in [11, Theorem 4.3.10].

(2) $\Rightarrow$ (3). Follows from definition of RD.

(3) $\Rightarrow$ (4). Suppose (4) is not true, then $I_p \subseteq I_r \subsetneq I_q$ for some $r \in S(L)$. By Proposition 2.5 (1), we have $I_p = I_q \cap K\{\bar{y}\} = I_r \cap K\{\bar{y}\}$ and r is an extension of p; but then by [6, Proposition 2 p.130] $\chi_q < \chi_r \leq \chi_p$ hence $\mathrm{RD}(q) < \mathrm{RD}(p)$.

(4) $\Rightarrow$ (5). Since RH can only go down under extension, we can assume L is a $|K|^+$-saturated differentially closed field. We prove by induction that

$$\mathrm{RH}(p) \geq \alpha \Rightarrow \mathrm{RH}(q) \geq \alpha.$$

The limit case is clear. Suppose $\mathrm{RH}(p) \geq \alpha + 1$. Then there exist $K' \supseteq K$ and $r, s \in S(K')$ such that r is an extension of p with $I_r \subsetneq I_s$ and $\mathrm{RH}(s) \geq \alpha$. Let L' be a differentially closed field containing both K' and L. Let $I_{s'}$ be an L'-component of I_s. By the induction hypothesis, $\mathrm{RH}(s') = \mathrm{RH}(s) \geq \alpha$. Let $I_{q'}$ be an L'-component of I_p contained in $I_{s'}$. The containment must be strict, otherwise $I_{q'}$ lies over $I_s \supsetneq I_r$ and therefore by Proposition 2.7 (differential polynomial ring version of the Going-Down theorem), $I_{q'}$ cannot be an L'-component of I_p. Thus $\mathrm{RH}(q') \geq \alpha+1$. Let $q'' = q'|_L$ then it follows from Proposition 2.7 again then $I_{q''} = I_{q'} \cap L\{\bar{y}\}$ must be an L-component of I_p. By Lemma 2.6, $I_{q''}$ and I_q are conjugates hence so are q'' and q. Therefore,

$$\mathrm{RH}(q) = \mathrm{RH}(q'') \geq \mathrm{RH}(q') \geq \alpha + 1.$$

(5) $\Rightarrow$ (2). Let I_r be an L-component of I_p containing in I_q. In particular, r extends p. If $I_r \subsetneq I_q$, then $\mathrm{RH}(q) < \mathrm{RH}(r) \leq \mathrm{RH}(p)$ contradicting (5). This shows that I_q is an L-component of I_p thus $\chi_q = \chi_p$ by (3) of Proposition 2.5. $\square$

If L is differentially closed, then by the differential nullstellensatz, one can replace condition (4) by "$V_L(I_q)$ is an L-irreducible component of $V_L(I_p)$". The main part of the following proof is taken from the proof of [4, Proposition 0.1.3], again the argument given there works equally well in the several derivations case.

Proposition 2.9. *Let $p \in S(K)$, if K is ω-saturated, then* $\Delta\text{-}\dim I_p = \mathrm{RH}(p)$.

Proof. By (4) of Proposition 2.3, it suffices to show that $\Delta\text{-}\dim I_p \geq \mathrm{RH}(p)$. The limit case is immediate. Suppose $\mathrm{RH}(p) \geq \alpha+1$ then there exist q, r over, L, some extension of K such that q extends p and $I_q \subsetneq I_r$ with $\mathrm{RH}(r) \geq \alpha$. By taking nonforking extensions if necessary, we can assume L is ω-saturated. Suppose b is a tuple from L consisting of the coefficients of a basis of I_r, say $\phi(y, b)$ is a conjunction of differential polynomial equations defining $V(I_r)$. Here we simply say that b is a tuple of parameters of I_r. Let a be a tuple of parameters of I_p from K. Then it is easy to see that

1. "$V(I_p) \supsetneq V(I_r)$" is expressible by a formula in $tp(b/a)$ and,
2. "$V(I_r)$ is irreducible" is expressible by an infinite collection of formulas in $tp(b)$.

By ω-saturation, let b' be a tuple in K realizing the type $tp(b/a)$. Hence by (2) the realizations of $\phi(y, b')$ in K and L are irreducible closed sets. By differential nullstellensatz, they determine prime differential ideals and hence types s' and s over L and K respectively with s' extending s. By condition (1) and the differential nullstellensatz, we have $I_s \supsetneq I_p$. Since b and b' have the same type over a and L is ω-saturated, therefore b and b' hence r and s' are conjugates under an automorphism of L. Therefore, $\mathrm{RH}(s) \geq \mathrm{RH}(s') = \mathrm{RH}(r) \geq \alpha$. Hence $\Delta\text{-}\dim I_p > \Delta\text{-}\dim I_s \geq \alpha$ by induction hypothesis. □

Finally we show that in m-DCF_0 Δ-dimension and RH are the same for complete types.

Proposition 2.10. $\Delta\text{-}\dim p = \mathrm{RH}(p)$

Proof. By definition, for every $\beta < \Delta\text{-}\dim p$, there exists some prime differential ideal P lying over I_p such that $\Delta\text{-}\dim P \geq \beta + 1$. Since $P = I_{p'}$ for some p' extending p, therefore by (2) and (4) of Proposition 2.3,

$$\beta + 1 \leq \Delta\text{-}\dim I_{p'} \leq \mathrm{RH}(p') \leq \mathrm{RH}(p).$$

We now show the reverse inequality by induction. Again the limit case is easy. Suppose $\mathrm{RH}(p) \geq \alpha + 1$. Then there are types q, r over some $L \supseteq K$ such that q extends p, $I_q \subsetneq I_r$ and $\mathrm{RH}(r) \geq \alpha$. Let L' be an ω-saturated extension of L and r' be a nonforking extension of r over L'. Let q' be a type over L' such that $V(I_{q'})$ is an L'-irreducible component of $V(I_q)$ containing $V(I_{r'})$. Hence $I_{q'} \subsetneq I_{r'}$, by Proposition 2.9 and Proposition 2.8 we have

$$\Delta\text{-}\dim p \geq \Delta\text{-}\dim I_{q'} > \Delta\text{-}\dim I_{r'} = \mathrm{RH}(r') = \mathrm{RH}(r) \geq \alpha.$$

This concludes the proof. □

3 Rank Inequalities

Let us recall the theorem that we want to prove.

Theorem 3.1. *Let p be a complete n-type in m-DCF_0 over a differential field K. Suppose d and e are the typical differential dimension and the differential type of p respectively and that $e \geq 1$. Then $1 \leq d \leq n$ and*

$$\omega^e d \leq \mathrm{RU}(p) \leq \mathrm{RM}(p) \leq \mathrm{RH}(p) \leq \mathrm{RD}(p) < \omega^e(d+1). \qquad (*)$$

Proof. Most of the work has been done in [11]. We have proved in Proposition 2.10 that Δ-dim = RH. The inequality $\mathrm{RD}(p) < \omega^e(d+1)$ is clear from the sequence of minimizing coefficient approach (§ 2). Therefore, it remains to show that $\mathrm{RH}(p) \leq \mathrm{RD}(p)$. We prove this by induction. The limit case is clear. Suppose $\mathrm{RH}(p) \geq \alpha + 1$. Then there are types q and r with q an extension of p and $I_q \subsetneq I_r$ and $\mathrm{RH}(r) \geq \alpha$. By the induction hypothesis, $\mathrm{RD}(r) \geq \alpha$. Since $I_q \subsetneq I_r$, $\chi_q > \chi_r$ [6, Proposition 2, p.130] and hence $\mathrm{RD}(q) > \mathrm{RD}(r)$. By (2) of Proposition 2.5, $\mathrm{RD}(r) < \mathrm{RD}(q)$ so we have

$$\mathrm{RD}(p) \geq \mathrm{RD}(q) > \mathrm{RD}(r) \geq \alpha.$$

This completes the proof. □

Examples 3.2. We conclude this article with some known examples in DCF_0 showing that each of the inequalities in Theorem 3.1 can be strict. These can certainly be viewed as evidence in supporting the claim made by Sack in the introduction of [18]: the least misleading example of a totally transcendental theory is the theory of differentially closed fields of characteristic 0.

1. It is easy to see that the first inequality can be strict. For example, let a be differentially transcendental over $\mathbb{Q}$ and b be a generic constant over $\mathbb{Q}(a)$. Then $tp(a, b)$ has U-rank $\omega + 1$. However, both its differential type and typical differential dimension are 1.

2. Let a be differentially transcendental over $\mathbb{Q}$ and E_a be the elliptic curve with j-invariant a. Let b be a realization of the generic type (over a) of the Manin kernel of E_a. Then $tp(a, b)$ has U-rank ω but Morley rank $\omega + 1$. This example is due to Hrushovski, see [17, addendum] for details. In [5], Hrushovski and Scanlon showed that in DCF_0 the Morley rank can be strictly greater than the U-rank even for types of finite Morley rank.

3. The type of a generic solution to the equation $y\delta^2 y - \delta y = 0$ has Morley rank 1 and differential height 2. This example is due to Poizat, for details see [8, p.64].

4. The type of a generic solution of the Painlevé equation $\delta^2 y = 6y^2 + a$ where a is an element such that $\delta a = 1$ has differential height 1 and differential order 2. The analysis of this Painlevé equation is due to Kolchin, see [8, p.66] for details.

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