ORDINAL DIMENSIONS AND DIFFERENTIAL COMPLETENESS

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THESIS

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To those who have given me their precious love!

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SUMMARY

We study the theory of ordinal dimensions and the notion of completeness in differential algebraic geometry. The techniques we use come from both model theory and differential algebra. In the course of understanding the complete sets in differential fields, we compute the formula for the Manin homomorphism of the generic family of elliptic curves. This formula and an application are given in Appendix B.

Dimension theory in differential algebraic geometry

Model theory provides several notions of ordinal dimension on definable sets in differentially closed fields. Morley rank (RM) Lascar rank (RU) and the continuous rank (RC) behave much better than the field theoretic (cardinal) dimension. Using this dimension theory, we show that: *every finite rank subset of a projective space must be affine. Also the restriction of a generic projection to a finite rank set must be injective.*

A fundamental question concerning ranks is whether Morley rank and Lascar rank are equal in differentially closed fields. Using Kolchin's dimension polynomial, we are able to give a useful upper bound of Morley rank. The main result here is: *for every definable set X, if* $\mathrm{RM}(X)$ *is a limit ordinal, then* $\mathrm{RM}(X) = \mathrm{RU}(X)$. Later Hrushovski and Scanlon showed that RU and RM are not in general equal in theory of differentially closed fields. We use this to show that RC *and* RM *differs in this theory also.*

In studying these various notion of dimensions, we come across the Kolchin Catenary Problem. *We give a positive answer to a special case of the problem: when the δ-variety we considered is actually an algebraic variety.*

Differential Completeness

Completeness of a variety is one of the most fundamental notion in algebraic geometry. Kolchin formulated the analogous concept in differential algebra. Based on a result in Model theory, we obtain a *Valuative Criterion for differential completeness.* Using this criterion, we give a proof of a result of Kolchin: The set of constant points in a projective space is differentially complete. Moreover, we are able to use this criterion to find a new family of differentially complete sets. Some members of this family are orthogonal to the field of constants. Hence from a Model Theoretic point of view they are very different from Kolchin's example. Furthermore, using the results we obtained in studying the ordinal dimensions, we show that *all the differentially complete sets are affine which is exactly the "opposite" phenomenon to that in algebraic geometry.*

CHAPTER 1

INTRODUCTION

Two areas in differential algebraic geometry will be studied in this paper: the theory of ordinal dimension and the notion of completeness. We will see how the interplay between model theory and algebra helps us to understand the geometric properties of various objects definable over a differential field.

First let us summarize here some of the definitions and facts that we need in the sequel. Our main reference for the model theory of differentially closed fields is [24]. We will assume some basic results in stability theory. A sound knowledge of Chapter 4 of [21] will be enough. Chapter 2 of [4] is a convenient reference for many results in differential algebra.

1.1 Definitions and basic results from Differential Algebra

By a **differential ring** we mean a pair $(\mathcal{R}, \delta)$ where $\mathcal{R}$ is a commutative ring containing the field, $\mathbb{Q}$, of rational numbers with 1 as the identity and $\delta : \mathcal{R} \to \mathcal{R}$ is a derivation on $\mathcal{R}$. That means δ is a group homomorphism of the additive group of $\mathcal{R}$ which also satisfies the Leibniz rule: $\delta(xy) = x\delta y + y\delta x$ for all $x, y \in \mathcal{R}$. In fact, what we have defined is also known as a **Ritt ring** in literature. Since we will use δ to denote the derivation for most of the time, we will simply call $\mathcal{R}$ a **δ-ring**. Similar abbreviations apply to other differential objects.

A δ-**ideal** of a δ-ring $\mathcal{R}$ is an ideal $\mathcal{I}$ of $\mathcal{R}$ such that $\delta\mathcal{I} \subseteq \mathcal{I}$. We call a δ-ideal **perfect** if it is equal to its radical. Also for any subset S of $\mathcal{R}$, we use $(S), [S], \{S\}$, to denote the smallest ideal, the smallest δ-ideal and the smallest perfect δ-ideal of $\mathcal{R}$ containing S.

A δ-**field** is a δ-ring which is also a field. If $\mathcal{F}$ is a δ-field, then the set of **constants**, $\mathcal{C} := \{x \in \mathcal{F} : \delta(x) = 0\}$ is a field also. Moreover, if $\mathcal{F}$ is algebraically closed, then so is $\mathcal{C}$.

Our assumption that δ-rings are $\mathbb{Q}$-algebras has an important consequence:

Proposition 1.1.1. *A δ-ring must have at least one prime δ-ideal.*

Proof. Let $\mathcal{R}$ be a δ-ring. Since $\mathcal{R} \supseteq \mathbb{Q}$, the radical of a δ-ideal in $\mathcal{R}$ is still a δ-ideal (see [17], p.62 or (1.3–1.6) in [4]). So the nilradical of $\mathcal{R}$ is a proper perfect δ-ideal in $\mathcal{R}$. By (1.6) in [4], every perfect δ-ideal in $\mathcal{R}$ is an intersection of prime δ-ideals. Therefore the result follows. □

By a δ-**homomorphism** we will understand a ring homomorphism $\sigma : \mathcal{R} \to \mathcal{S}$ between δ-rings commuting with the derivation. That is $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in \mathcal{R}$. In this case, we will say $\mathcal{S}$ **is a** δ-$\mathcal{R}$-**algebra**.

Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are two δ-$\mathcal{R}$-algebras. If the tensor product $\mathcal{S}_1 \otimes_{\mathcal{R}} \mathcal{S}_2$ is nonzero, then $q \mapsto q \otimes 1 = 1 \otimes q$ is a nonzero ring homomorphism from $\mathbb{Q}$ into the tensor hence is an embedding. In this case, $\mathcal{S}_1 \otimes_{\mathcal{R}} \mathcal{S}_2$ is a δ-ring under that natural derivation: $\delta(s_1 \otimes s_2) = \delta s_1 \otimes s_2 + s_1 \otimes \delta s_2$.

If σ is an inclusion, then we say that $\mathcal{S}$ is a δ-**ring extension** (or simply an extension) of $\mathcal{R}$ and $\mathcal{R}$ is a δ-**subring** of $\mathcal{S}$. In this case, we regard $\mathcal{R}$ as a subring of $\mathcal{S}$ via σ. Let A be a subset of $\mathcal{S}$. The δ-$\mathcal{R}$-**algebra generated by** A is the smallest δ-subring of $\mathcal{S}$ containing both

A and $\sigma(\mathcal{R})$. We denote this δ-ring by $\mathcal{R}\{A\}$. If $\mathcal{S}$ is a δ-field and $\mathcal{R}$ is a δ-subfield of $\mathcal{S}$ then we use $\mathcal{R}\langle A\rangle$ to denote the **δ-field generated by A over $\mathcal{R}$.**

Let $\mathcal{F}$ be a δ-field and $\mathcal{G}$ be a δ-field extension of $\mathcal{F}$. We say that a subset B of $\mathcal{G}$ is **δ-algebraically independent (respectively dependent) over $\mathcal{F}$** if $\bigcup_{r\geq 0}\delta^r B$, the set of derivatives of B, is an algebraically independent (respectively dependent) set over $\mathcal{F}$. We define the **δ-transcendence degree of $\mathcal{G}$ over $\mathcal{F}$** to be

$$\mathrm{td}_\delta(\mathcal{G}/\mathcal{F}) = \max\{|B| : B \subset \mathcal{G} \ \delta\text{-algebraically independent over } \mathcal{F}.\}$$

Let $\mathcal{F}$ be a δ-field. The **δ-polynomial ring generated by $y_1, \dots, y_n$ over $\mathcal{F}$** is the polynomial ring

$$\mathcal{F}\{y_1, \dots, y_n\} := \mathcal{F}[\delta^r y_i : r \geq 0, \ i = 1, \dots, n].$$

We call elements of $\mathcal{F}\{y_1, \dots, y_n\}$ **δ-polynomials over $\mathcal{F}$**. It is easy to see that $\mathcal{F}\{y_1, \dots, y_n\}$ is not a Noetherian ring, however a version of Hilbert Basis Theorem does hold for the collection of perfect δ-ideals.

Theorem 1.1.2 (Differential Basis Theorem). *For any perfect δ-ideal $\mathcal{I}$ in $\mathcal{F}\{y_1, \dots, y_n\}$, there is some finite subset S of $\mathcal{I}$ such that $\mathcal{I} = \{S\}$.*

We say that a δ-field $\mathcal{F}$ is **differentially closed** if for any $n \geq 1$ and any prime δ-ideal $\mathcal{P} \subset \mathcal{F}\{y_1, \dots, y_n\}$, $\mathcal{P}$ has a zero in $\mathcal{F}^n$.

1.2 The Kolchin Topology

The basic geometric objects that we are interested in are the δ-closed subsets of the affine and the projective spaces.

We say a subset X of the affine space $\mathbb{A}^n$ is **δ-closed** if it is the zero set of a collection of δ-polynomials in $\mathcal{F}\{y_1, \ldots, y_n\}$. By the differential basis theorem, there is a finite subcollection which defines the same set. Therefore an affine δ-closed subset of $\mathbb{A}^n$ is of the form

$$X = \{\bar{x} \in \mathbb{A}^n : f_1(\bar{x}) = \cdots = f_m(\bar{x}) = 0\}$$

where $f_1, \ldots, f_m \in \mathcal{F}\{t_1, \ldots, t_n\}$.

The δ-closed subsets of $\mathbb{A}^n$ form a topology. We call this the **Kolchin topology** on $\mathbb{A}^n$. In general, the **Kolchin topology on an $\mathcal{F}$-variety** V is the topology on V induced locally by the Kolchin topology on the affine open subsets of V. A subset X of V is **δ-irreducible** if X (with the induce δ-topology) cannot be written as a union of two proper δ-closed subsets.

By a **projective (respectively affine) δ-variety** we will understand a δ-irreducible δ-closed subset of some projective space (respectively affine space). A **quasiprojective (respectively quasiaffine) δ-variety** is a δ-open subset of a projective (respectively affine) δ-variety. We will simply say that X is a δ-variety when we mean X is a quasiprojective δ-variety.

Let Σ be an affine δ-closed set and $\mathcal{I} := \{f \in \mathcal{F}\{y_1, \ldots, y_n\} : f(\bar{x}) = 0 \ \forall \bar{x} \in \Sigma\}$ be the perfect δ-ideal corresponding to Σ. The **δ-coordinate ring of** Σ, denoted by $\mathcal{F}\{\Sigma\}$, is defined to be the δ-ring $\mathcal{F}\{y_1, \ldots, y_n\}/\mathcal{I}$. If Σ is δ-irreducible then $\mathcal{I}$ is a prime δ-ideal. In this case, we define the **field of δ-rational functions on** Σ to be, $\mathcal{F}\langle\Sigma\rangle$, the field of fractions of $\mathcal{F}\{\Sigma\}$. If Σ is not affine the above concepts are best defined using the sheaf of δ-polynomial functions on Σ. We refer the readers to [4] p.55-56 for their definitions.

Let $d \in \mathbb{N}$, we say that a non-constant δ-polynomial $f \in \mathcal{F}\{y_0, \dots, y_n\}$ is **δ-homogeneous of degree** d if for some (for any) t δ-indeterminate over $\mathcal{F}\{y_0, \dots, y_n\}$

$$f(ty_0, \dots, ty_n) = t^d f(y_0, \dots, y_n)$$

as elements in the δ-polynomial ring $\mathcal{F}\{y_0, \dots, y_n, t\}$.

Just like ordinary homogeneous polynomials, δ-homogeneous polynomials can be obtained by homogenization. Let f be a δ-polynomial in $y_1, \dots, y_n$. One can easily check that, for d sufficiently large, $y_0^d f(\frac{y_1}{y_0}, \dots, \frac{y_n}{y_0})$ is a δ-homogeneous polynomial of degree d. For example by homogenizing the δ-polynomial $\delta(y_1)$, we get $y_0^2\delta(\frac{y_1}{y_0}) = y_0\delta(y_1) - y_1\delta(y_0)$ which is a δ-homogeneous polynomial of degree 2.

Using δ-homogeneous polynomials, one can give a more direct characterization of δ-closed subsets of $\mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{A}^m$.

- A subset of $\mathbb{P}^n$ is **δ-closed** if it is the zero set of a collection of δ-homogeneous polynomials in $\mathcal{F}\{y_0, \dots, y_n\}$.
- A subset of $\mathbb{P}^n \times \mathbb{A}^m$ is **δ-closed** if it is the zero set of a collection of δ-polynomials $\{f_i\}$ in $\mathcal{F}\{y_0, \dots, y_n, z_1, \dots, z_m\}$ such that each f_i is δ-homogeneous in the valuables $y_0, \dots, y_n$.

Example 1.2.1. Let Z be the δ-closed subset of $\mathbb{A}^2$ defined by:

$$z\delta(y)^2 + (y^4 - 1) = 0$$

$$2z\delta^2(y) + \delta(z)\delta(y) + 4y^3 = 0.$$

Homogenizing the first equation with respect to y we get

$$z(y_1\delta(y_0) - y_0\delta(y_1))^2 + y_0^4 - y_1^4 = 0. \tag{1.1}$$

Note that in this example, the δ-closed set defined by Equation 1.1 does not intersect $[1:0]\times\mathbb{A}^1$. Hence Z can be thought as a δ-closed subset of $\mathbb{P}^1 \times \mathbb{A}^1$.

Further discussion of differential algebra and Kolchin topology can be found in [17].

1.3 Basic Model Theory of Differentially Closed Fields

Let $\mathfrak{L}_\delta = \{0, 1, +, -, \cdot, \delta\}$ be the language of rings together with an extra function symbol δ. In our discussion, δ will always be interpreted as a derivation. The first order theory of a differentially closed field of characteristic 0 in this language is complete. We will use DCF_0 to denote this complete theory.

From now on, $\mathcal{U}$ will be a fixed saturated model of DCF_0. All the δ-fields that appear are δ-subfields of $\mathcal{U}$ with cardinality smaller than the cardinality of $\mathcal{U}$.

In DCF_0, there is an algebraic characterization of forking. Let $\mathcal{G}, \mathcal{H} \supseteq \mathcal{F}$ be δ-fields. We say that $\mathcal{G}$ **does not fork from** $\mathcal{H}$ **over** $\mathcal{F}$, write as $\mathcal{G} \perp_{\mathcal{F}} \mathcal{H}$, if $\mathcal{G}$ and $\mathcal{H}$ are algebraically disjoint (as fields) over $\mathcal{F}$. Let A, B and C be subsets of $\mathcal{U}$. Later on when we write $A \mathop{\smile\!\!\!\!\!|}_C B$, we mean $\mathbb{Q}\langle AC\rangle \mathop{\smile\!\!\!\!\!|}_{\mathbb{Q}\langle C\rangle} \mathbb{Q}\langle BC\rangle$. If C is the empty set, we simply write $A \mathop{\smile\!\!\!\!\!|} B$.

With the notion of forking, we can define the **Lascar rank (or the U-rank)**, RU, inductively as follows:

Definition 1.3.1. Let $\bar{a}$ be a tuple from $\mathcal{U}$ and $B \subset \mathcal{U}$.

- $\mathrm{RU}(\bar{a}/B) \geq 0$.
- If α is a limit ordinal, then $\mathrm{RU}(\bar{a}/B) \geq \alpha$ if $\mathrm{RU}(\bar{a}/B) \geq \beta$, for all $\beta < \alpha$.
- $\mathrm{RU}(\bar{a}/B) \geq \alpha + 1$ if there exists $C \supset B$ such that $\mathrm{RU}(\bar{a}/C) \geq \alpha$ and $\bar{a} \not\mathop{\smile\!\!\!\!\!|}_B C$.

We say that $\mathrm{RU}(\bar{a}/B)$ **is equal to** α if $\mathrm{RU}(\bar{a}/B) \geq \alpha$ but $\mathrm{RU}(\bar{a}/B) \not\geq \alpha + 1$.

Let $p \in S_n(A)$ be an n-type over A. The **U-rank of** p, $\mathrm{RU}(p)$, is defined to be $\mathrm{RU}(\bar{a}/A)$ where $\bar{a}$ is any realization of p in $\mathcal{U}^n$. Let B be an A-definable set; **we define** $\mathrm{RU}(B/A)$ **to be** $\sup\{\mathrm{RU}(b/A) : b \in B\}$. Note that this definition is independent of A. Suppose B is defined over some other A'. Let $C = A \cup A'$, it is clear that $\mathrm{RU}(B/C) \leq \mathrm{RU}(B/A)$. But since every type over A has a nonforking extension to C. Therefore $\mathrm{RU}(B/C) = \mathrm{RU}(B/A)$. Replacing A by A', we see that $\mathrm{RU}(B/C) = \mathrm{RU}(B/A')$ as well. We will argue that $\mathrm{RU}(B)$ can always be attained by some $b \in B$ (Chapter 2 Proposition 2.2.6).

Lascar rank gives a natural dimension on definable sets. It enjoys the following nice properties:

1. $\mathrm{RU}(a/B) = 0$ if and only if a is algebraic over $\mathbb{Q}\langle B\rangle$.

2. $\mathrm{RU}(a/B) = \omega$ if and only if a is δ-transcendental over $\mathbb{Q}\langle B\rangle$.

3. RU is invariant under definable bijection.

Moreover, RU satisfies the Lascar inequalities:

$$\mathrm{RU}(\bar{a}/A, \bar{b}) + \mathrm{RU}(\bar{b}/A) \leq \mathrm{RU}(\bar{a}, \bar{b}/A) \leq \mathrm{RU}(\bar{a}/A, \bar{b}) \oplus \mathrm{RU}(\bar{b}/A) \tag{1.2}$$

where $+$ is the usual ordinal sum and $\oplus$ is the Cantor's symmetric sum of ordinals. For a proof, see [20] or [32].

As an example, let us calculate $\mathrm{RU}(\mathbb{P}^n)$ and $\mathrm{RU}(\mathbb{A}^n)$. As a set, $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ where $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\}$ is isomorphic to $\mathbb{A}^n$ for each i. Therefore, $\mathrm{RU}(\mathbb{P}^n) = \mathrm{RU}(\mathbb{A}^n)$. By saturation, there is an $a \in \mathcal{U}$ which is δ-transcendental over $\mathbb{Q}$. We have $\mathrm{RU}(a) = \mathrm{RU}(\mathbb{A}^1) = \omega$. By induction, there exists $(a_1, \ldots, a_{n-1}) \in \mathcal{U}^{n-1}$ such that $\mathrm{RU}(\mathbb{A}^{n-1}) = \mathrm{RU}(a_1, \ldots, a_{n-1}) = \omega(n-1)$. By saturation again, there is an $a_n \in \mathcal{U}$ such that a_n is δ-transcendental over the field generated by $a_1, \ldots, a_{n-1}$. Since $\omega + \omega(n-1) = \omega n = \omega \oplus \omega(n-1)$, the ends of the Lascar inequalities are equal. Therefore $\mathrm{RU}(\mathbb{A}^n) = \omega n$.

Next we are going to define Morley rank. To illustrate its geometric nature, we will define it on the definable sets in a saturated structure (e.g. in $\mathcal{U}$).

Definition 1.3.2. Let M be an infinite saturated first order structure and X be a definable subset of M^n (with parameters from M). We define Morley rank of X, $\mathrm{RM}(X)$ as follows.

- $\mathrm{RM}(X) \geq 0$ if X is nonempty. $\mathrm{RM}(\emptyset) = -1$.

- If α is a limit ordinal, then $\mathrm{RM}(X) \geq \alpha$ if $\mathrm{RM}(X) \geq \beta$ for all $\beta < \alpha$.
- $\mathrm{RM}(X) \geq \alpha + 1$ if there exists Y_i $(i < \omega)$ nonempty pairwise disjoint definable subsets of X such that $\mathrm{RM}(Y_i) \geq \alpha$ for each $i < \omega$.

We say that $\mathrm{RM}(X) = \alpha$ **if** $\mathrm{RM}(X) \geq \alpha$ **but** $\mathrm{RM}(X) \not\geq \alpha + 1$. If no such α exists, then the Morley rank of X is underfined and we write $\mathrm{RM}(X) = \infty$. The **Morley rank of a formula** φ is defined to be $\mathrm{RM}(X)$ where X is the subset of M defined by φ. Let A be a subset of M, the **Morley rank of a type** p **over** A is defined to be $\inf\{\mathrm{RM}(\varphi) : \varphi \in p\}$. The **Morley rank of a tuple** $\bar{a}$ **over** A, denoted by $\mathrm{RM}(\bar{a}/A)$, is defined to be the Morley rank of the type of $\bar{a}$ over A.

Another important concept that we will encounter is orthogonality.

Definition 1.3.3. Let p_i be a type over A_i $(i = 1, 2)$. We say that p_1 **is orthogonal to** p_2, written as $p_1 \perp p_2$, if for any $B \supseteq A_1 \cup A_2$ and $\bar{c}_i$ realization of p_i such that $\bar{c}_i \mathop{\smile\!\!\!\!\!\mid}_{A_i} B$, we have $c_1 \mathop{\smile\!\!\!\!\!\mid}_B c_2$.

Let D be a subset (possibly empty) of $\mathcal{U}$. We say that **a type** p **is orthogonal to** D if $p \perp q$ for every type q over D.

Let φ_i be a strongly minimal formula $(i = 1, 2)$. We say that φ_1 **is orthogonal to** φ_2 if for any p_i of Morley rank 1 containing φ_i, $p_1 \perp p_2$.

Finally we list the facts about DCF_0 that we are going to use. They can be found in [24] and [30].

1. DCF_0 has quantifier elimination in $\mathfrak{L}_\delta$.

2. DCF_0 has elimination of imaginaries in $\mathfrak{L}_\delta$.

3. DCF_0 is ω-stable.

4. Let $X \subset \mathcal{U}^n$ be a set definable in the pure language of rings (with parameters). Let $\mathrm{RU}^-(X)$ (respectively $\mathrm{RM}^-(X)$) be the Lascar rank (respectively the Morley rank) of X calculated as a definable subset of $\mathcal{U}^n$ when we regard $\mathcal{U}$ simply as an algebraically closed field (forgetting the differential field structure on $\mathcal{U}$). Then we have $\mathrm{RU}(X) = \omega\,\mathrm{RU}^-(X)$ and $\mathrm{RM}(X) = \omega\,\mathrm{RM}^-(X)$. Moreover, in the case when X is an algebraic variety, we have $\mathrm{RU}^-(X) = \mathrm{RM}^-(X) = \dim(X)$.

For other model-theoretic notions used in this section, readers can consult [10] and [21].

CHAPTER 2

ORDINAL DIMENSIONS IN DIFFERENTIALLY CLOSED FIELDS

It is well-known that in the theory of algebraically closed field, Morley rank (RM) is equal to Lascar rank. In DCF_0, as in any ω-stable theory, we know that $\mathrm{RU} \leq \mathrm{RC} \leq \mathrm{RM}$, where RC is the continuous rank [21][Chapter 4]. The rank (in any of the above senses) of an element a over a set B is always less than or equal to ω. It is equal to ω if and only if a is δ-transcendental over $\mathbb{Q}\langle B\rangle$. In this chapter, we are going to study the relations between these ranks in DCF_0. We show that if the Morley rank of a type p is a limit ordinal, then $\mathrm{RU}(p) = \mathrm{RM}(p)$. From this we deduce that $\mathrm{RM} = \mathrm{RC}$ implies $\mathrm{RM} = \mathrm{RU}$ in DCF_0. Since our proof, Hrushovski and Scanlon have shown that RM and RU differ in DCF_0 (see Appendix A). Therefore we also have $\mathrm{RM} \neq \mathrm{RC}$ in DCF_0.

2.1 The Kolchin Polynomials

We start by reviewing some notions of differential algebra. Let $\mathcal{R}$ be a δ-ring. Denote by $\mathrm{Spec}_\delta\,\mathcal{R}$ the set of all prime δ-ideals in $\mathcal{R}$. Together with the induced Zariski topology, $\mathrm{Spec}_\delta\,\mathcal{R}$ can be regarded as a subspace of $\mathrm{Spec}\mathcal{R}$. Moreover if $\mathrm{Spec}_\delta\,\mathcal{R}$ is Noetherian, then every prime δ-ideal of $\mathcal{R}$ has a natural ordinal dimension.

Definition 2.1.1. Let $\mathcal{P} \in \mathrm{Spec}_\delta\,\mathcal{R}$. The **$\delta$-dimension** of $\mathcal{P}$ is defined inductively as:

- $\dim_\delta \mathcal{P} = 0$ if $\mathcal{P}$ is a maximal element in $\mathrm{Spec}_\delta\,\mathcal{R}$;
- $\dim_\delta \mathcal{P} = \sup\{\dim_\delta \mathcal{Q} + 1 : \mathcal{Q} \in \mathrm{Spec}_\delta\,\mathcal{R}, \mathcal{Q} \supsetneq \mathcal{P}\}$.

Furthermore, if $\mathcal{R}$ is also an integral domain, then we define the **δ-dimension of $\mathcal{R}$,** $\dim_\delta \mathcal{R}$ to be the δ-dimension of the zero ideal of $\mathcal{R}$.

Let $\mathcal{K}$ be a differentially closed field. To each type $p \in S_n(\mathcal{K})$, we associate a prime δ-ideal

$$\mathcal{I}_p = \{f(\bar{t}) \in \mathcal{K}\{t_1, \dots, t_n\} : \text{“}f(\bar{t}) = 0\text{”} \in p\}.$$

This is a one to one correspondence between $S_n(\mathcal{K})$ and $\operatorname{Spec}_\delta \mathcal{K}\{t_1, \dots, t_n\}$. If $\bar{a}$ is a realization of p, then $\mathcal{I}_p = \operatorname{Ker}\varphi_{\bar{a}}$ where $\varphi_{\bar{a}}$ is the δ-$\mathcal{K}$-algebra homomorphism from $\mathcal{K}\{t_1, \dots, t_n\}$ to $\mathcal{U}$ determined by sending t_i to a_i, for $i = 1, \dots, n$.

Morley rank and δ-dimension are related. Our next result shows that Morley rank is bounded above by δ-dimension.

Lemma 2.1.2. *Let $\mathcal{L}$ be a δ-field. Let $\mathcal{K}$ be an ω-saturated differentially closed extension of $\mathcal{L}$. Let $p \in S_n(\mathcal{L})$ and $\tilde{p} \in S_n(\mathcal{K})$ be a nonforking extension of p on $\mathcal{K}$. Then* $\mathrm{RM}(p) \leq \dim_\delta I_{\tilde{p}}$.

Proof. Since $\tilde{p}$ is a nonforking extension of p, $\mathrm{RM}(p) = \mathrm{RM}(\tilde{p})$. It suffices to show that **for any** $q \in S_n(\mathcal{K})$**, if** $\mathrm{RM}(q) \geq \alpha$ **then** $\dim_\delta \mathcal{I}_q \geq \alpha$. Since δ-dimension of a prime δ-ideal is always nonnegative, the implication clearly holds for the case $\alpha = 0$. When α is a limit ordinal, the implication follows easily from the induction hypothesis. So suppose $\mathrm{RM}(q) \geq \alpha + 1$. Let $f_1, \dots, f_k$ be a basis of $\mathcal{I}_q$ then the formula $\bigwedge_{i=1}^k f_i(\bar{t}) = 0$ is in q. By the definition of Morley rank, (here we use the fact that Morley rank is equal to Cantor-Bendixson rank in ω-saturated structures, (see Lemma 5.6.3 in [10])) there exists $r \neq q$ in $S_n(\mathcal{K})$ such that the formula $\bigwedge_{i=1}^{k} f_i(\bar{t}) = 0$ is in r and $\mathrm{RM}(r) \geq \alpha$. So we have $\mathcal{I}_q \subsetneq \mathcal{I}_r$. By the definition of δ-dimension and the induction hypothesis we have $\dim_\delta \mathcal{I}_q > \dim_\delta \mathcal{I}_r \geq \alpha$. □

Using the correspondence between prime δ-ideals of $\mathcal{K}\{t_1, \dots, t_n\}$ which contain $\mathcal{I}_{\tilde{p}}$ and prime δ-ideals of $\mathcal{K}\{t_1, \dots, t_n\}/\mathcal{I}_{\tilde{p}} \cong \mathcal{K}\{\bar{a}\}$, we can rewrite the conclusion of Lemma 2.1.2 as $\mathrm{RM}(p) \leq \dim_\delta \mathcal{K}\{\bar{a}\}$ where $\bar{a} \in \mathcal{U}^n$ is a realization of $\tilde{p}$.

The following results are inspired by the works of J.Johnson in [14] and [13]. He defined in [14] the Krull dimension of a δ-ring in a general setting. Since we are dealing with ordinary differential rings only (i.e. differential rings with only one derivation), no such level of generality is required. Nevertheless we encourage readers who are interested in the results in several derivations to consult the original papers.

Let $\mathcal{K}$ be a δ-field (not necessary δ-closed) and $\mathcal{R}$ be a δ-integral domain which is also a finitely generated δ-$\mathcal{K}$-algebra. By the differential basis theorem, $\mathrm{Spec}_\delta\, \mathcal{R}$ is a Noetherian topological space. We fix a set of generators $a_1, \dots, a_n$ for $\mathcal{R}$; let

$$R^{[r]} = \mathcal{K}[\delta^j a_i : 1 \leq i \leq n, 0 \leq j \leq r]$$

for $r \geq 0$ and $R^{[-1]} = \mathcal{K}$.

Lemma 2.1.3. *Let $\mathcal{L}/\mathcal{K}$ be a differential field extension. Let S be a subset of $\mathcal{L}$. If $b_1, \dots, b_k$ in $\mathcal{L}$ are algebraically dependent over $\mathcal{K}(S)$, then $\delta b_1, \dots, \delta b_k$ are algebraically dependent over the field $\mathcal{K}(S \cup \delta S \cup \{b_1, \dots, b_k\})$.*

Proof. Let $P(y_1, \dots, y_k)$ be a nonzero polynomial over $\mathcal{K}(S)$ with minimal total degree such that $P(b_1, \dots, b_k) = 0$. Apply δ to both sides of this equation and we get

$$P^\delta(\bar{b}) + \sum_{i=1}^{k} \frac{\partial P}{\partial y_i}(\bar{b})\delta b_i = 0 \tag{2.1}$$

where P^δ is the polynomial obtained by applying δ to the coefficients of P. Since $P \neq 0$, for some $1 \leq i \leq k$, $\frac{\partial P}{\partial y_i}$ is a nonzero polynomial which has total degree less that P. So by our choice of P, $\frac{\partial P}{\partial y_i}(\bar{b}) \neq 0$, therefore Equation 2.1 gives a nontrivial algebraic relation among the δb_i's over the field $\mathcal{K}(S \cup \delta S \cup \{b_1, \dots, b_k\})$.

Note that for the case $k = 1$, we can conclude that δb_1 is even algebraic over $\mathcal{K}(S \cup \delta S)$. This follows from the lemma and that b_1 itself is algebraic over $\mathcal{K}(S \cup \delta S)$. □

Let $L^{[r]}$ be the field of fractions of $R^{[r]}$ and $\mathcal{L}$ be the field of fractions of $\mathcal{R}$, note that $\mathcal{L}$ is a δ-field. For all sufficiently large r, the relation between r and the transcendence degree of $L^{[r]}$ over K is given by the following proposition.

Proposition 2.1.4. *There is a polynomial of the form $dt + b$, where d is the δ-transcendence degree of $\mathcal{L}/\mathcal{K}$ and b is a nonnegative integer, such that* $\operatorname{td}(L^{[r]}/\mathcal{K}) = dr + b$ *for all sufficiently large $r \in \mathbb{N}$.*

Proof. For any $r \geq 0$ we have $L^{[r]} = L^{[r-1]}(\delta^r a_1, \dots, \delta^r a_n)$. Renumbering the generators if necessary, we may assume $\delta^r a_1, \dots, \delta^r a_e$ forms a transcendence basis of $L^{[r]}/L^{[r-1]}$ (e can be 0). By Lemma 2.1.3, $\delta^{r+1} a_i$ is algebraic over $L^{[r]}(\delta^{r+1} a_1, \dots, \delta^{r+1} a_e)$ for every $i \geq e+1$. Since r is arbitrary, we see that $\operatorname{td}(L^{[r]}/L^{[r-1]})$ is a non-increasing function of r. Consequently there exist integers m and d such that $\operatorname{td}(L^{[r]}/L^{[r-1]}) = d \geq 0$, for all $r \geq m$. Also note that at each stage we add at least d to the transcendence degree; so $\operatorname{td}(L^{[m-1]}/\mathcal{K}) \geq md$. Thus for $r \geq m$, we have

$$\begin{aligned}\operatorname{td}(L^{[r]}/\mathcal{K}) &= \sum_{j=m}^{r} \operatorname{td}(L^{[j]}/L^{[j-1]}) + \operatorname{td}(L^{[m-1]}/\mathcal{K}) \\ &= (r-m+1)d + \operatorname{td}(L^{[m-1]}/\mathcal{K}) = dr + b\end{aligned}$$

for some $b \geq d$. It remains to argue that d is the δ-transcendence degree of $\mathcal{L}/\mathcal{K}$. To prove this, pick a transcendence basis of $L^{[m]}/L^{[m-1]}$. Without loss of generality, say it is $\{\delta^m a_1, \ldots, \delta^m a_d\}$. If $d = 0$, then the required polynomial is simply a constant. On one hand, it follows from Lemma 2.1.3 that $\delta^j a_1, \ldots, \delta^j a_d$ are algebraically independent over $L^{[j-1]}$, for all $0 \leq j \leq m$. On the other hand, by (2.1.3) again, we conclude that $\delta^{m+1} a_i$ is algebraic over $L^{[m]}(\delta^{m+1} a_1, \ldots, \delta^{m+1} a_d)$, for each $i \geq d+1$. But $\operatorname{td}(L^{[m+1]}/L^{[m]}) = d$ hence $\delta^{m+1} a_1, \ldots, \delta^{m+1} a_d$ must remain algebraically independent over $L^{[m]}$. By repeating the same argument, we conclude that $\{a_1, \ldots, a_d\}$ is a maximal δ-independent subset of the generators so $\operatorname{td}_\delta(\mathcal{L}/\mathcal{K}) = d$.

Thus $dt + b$ is the polynomial with the required property. We call this the **Kolchin polynomial** [1] **of $\bar{a}$ over $\mathcal{K}$** (or of $\mathcal{R}$ if the generators of $\mathcal{R}$ over $\mathcal{K}$ are clear). □

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[1] It is also known as the dimension polynomial. In general, it is a polynomial with degree less than or equal to the number of derivations. For details, see [13], [17] or [26].

2.2 Relations between Ranks

For any $\mathcal{P} \in \operatorname{Spec}_\delta \mathcal{R}$, let $a_\mathcal{P} t + b_\mathcal{P}$ be the Kolchin polynomial of $\mathcal{R}/\mathcal{P}$. If $\mathcal{Q}$ is another prime δ-ideal containing $\mathcal{P}$, then for $r \in \mathbb{N}$ sufficiently large, $P^{[r]} := \mathcal{P} \cap R^{[r]} \subset Q^{[r]} := \mathcal{Q} \cap R^{[r]}$. Since $R^{[r]}$ is finitely generated as a $\mathcal{K}$-algebra, $\operatorname{td}((R^{[r]}/P^{[r]})/\mathcal{K}) > \operatorname{td}((R^{[r]}/Q^{[r]})/\mathcal{K})$. Note also that $(\mathcal{R}/\mathcal{P})^{[r]}$ and $R^{[r]}/P^{[r]}$ are isomorphic as rings. Thus $a_\mathcal{P} r + b_\mathcal{P} = \operatorname{td}((\mathcal{R}/\mathcal{P})^{[r]}/\mathcal{K}) > \operatorname{td}((\mathcal{R}/\mathcal{Q})^{[r]}/\mathcal{K}) = a_\mathcal{Q} r + b_\mathcal{Q}$, for all large enough r. That simply means $\omega a_\mathcal{P} + b_\mathcal{P} > \omega a_\mathcal{Q} + b_\mathcal{Q}$ as ordinals. Using this observation, we can show that the δ-dimension of a prime δ-ideal is bounded above by the value of its Kolchin polynomial at ω.

Proposition 2.2.1. *For any* $\mathcal{P} \in \operatorname{Spec}_\delta \mathcal{R}, \dim_\delta \mathcal{P} \leq \omega a_\mathcal{P} + b_\mathcal{P}$.

Proof. We prove this by induction on the δ-dimension of $\mathcal{P}$. Let $\dim_\delta \mathcal{P} = \alpha$. By definition of δ-dimension, for any $\beta < \alpha$ there exists $\mathcal{Q} \in \operatorname{Spec}_\delta \mathcal{R}$ such that $\mathcal{Q} \supset \mathcal{P}$ and $\beta \leq \dim_\delta \mathcal{Q} < \alpha$. By the induction assumption and the above discussion, we have $\dim_\delta \mathcal{Q} \leq \omega a_\mathcal{Q} + b_\mathcal{Q} < \omega a_\mathcal{P} + b_\mathcal{P}$. Since $\beta < \alpha$ is arbitrary, we conclude that $\omega a_\mathcal{P} + b_\mathcal{P} \geq \alpha = \dim_\delta \mathcal{P}$. □

Corollary 2.2.2. $\dim_\delta \mathcal{R} < \omega(\operatorname{td}_\delta(\mathcal{R}/\mathcal{K}) + 1)$.

Proof. By Proposition 2.1.4, the Kolchin polynomial of the zero ideal is of the form $dt + b$ where $d = \operatorname{td}_\delta(\mathcal{R}/\mathcal{K})$. So by Proposition 2.2.1, we have $\dim_\delta \mathcal{R} \leq \omega d + b < \omega(d+1)$. □

Now we can prove the result promised at the beginning of this chapter.

Theorem 2.2.3. *Let* $p \in S_n(\mathcal{K})$*. If* $\operatorname{RM}(p)$ *is a limit ordinal, then* $\operatorname{RU}(p) = \operatorname{RM}(p)$.

Proof. By taking a nonforking extension of p, we can assume $\mathcal{K}$ is ω-saturated. Let $(a_1, \ldots, a_n)$ be a realization of p. Without loss of generality, $a_1, \ldots, a_d$ forms a δ-transcendence base of $\mathcal{K}\langle a_1, \ldots, a_n\rangle$. By induction and Lascar inequalities, we have $\mathrm{RU}(p) \geq \mathrm{RU}(a_1, \ldots, a_d/\mathcal{K}) = \omega d$. On the other hand by Lemma 2.1.2 and Corollary 2.2.2, $\mathrm{RM}(p) \leq \dim_\delta \mathcal{K}\{\bar{a}\} < \omega(d+1)$. So all together, we have

$$\omega d \leq \mathrm{RU}(p) \leq \mathrm{RM}(p) < \omega(d+1).$$

Since the last inequality is strict, if $\mathrm{RM}(p)$ is a limit, it must be equal to ωd. This forces $\mathrm{RU}(p) = \mathrm{RM}(p)$. □

The following is a general result about the various ranks that we have considered. The definition of RC is taken from [21].

Theorem 2.2.4. *Let T be a totally transcendental theory. If the following hold:*

1. *For any type p, if $\mathrm{RM}(p)$ is a limit ordinal, then $\mathrm{RM}(p) = \mathrm{RU}(p)$*

2. $\mathrm{RM} = \mathrm{RC}$

then $\mathrm{RM} = \mathrm{RU}$.

Proof. Let A be a subset of some model of T and p be an m type over A. We are going to show by induction on α that if $\mathrm{RM}(p) = \alpha$, then $\mathrm{RU}(p) = \alpha$.

When $\alpha = 0$, there is nothing to prove since p is algebraic. In this case, $\mathrm{RU}(p) = \mathrm{RM}(p) = 0$. Our assumption takes care of the case when α is a limit. Suppose $\mathrm{RM}(p) = \alpha + 1$, RM and

RU agree up to α. Since it is always the case that $\mathrm{RU}(p) \leq \mathrm{RM}(p)$, it suffices to show that $\mathrm{RU}(p) > \alpha$.

Pick $\varphi \in p$ such that $\mathrm{RM}(\varphi) = \mathrm{RM}(p) = \alpha+1$ and φ isolates p among those types in $S_m(A)$ which have Morley rank greater than or equal to $\alpha+1$. By assumption (2), $\mathrm{RC}(\varphi) = \alpha+1$ which means that there is a formula ψ with parameters in some $B \supseteq A$ such that $\mathrm{RC}(\psi) = \alpha$, ψ forks over A and $\forall \bar{x}[\psi(\bar{x}) \longrightarrow \varphi(\bar{x})]$. Choose q in $S_n(B)$ containing ψ such that $\mathrm{RC}(q) = \mathrm{RC}(\psi)$. Now by the induction hypothesis, $\mathrm{RM}(q) = \mathrm{RU}(q) = \alpha$. Since ψ forks over A, we have $\mathrm{RM}(q|_A) \geq \mathrm{RU}(q|_A) \geq \alpha+1$. Moreover, we have $\varphi \in q|_A$. Therefore by our choice of φ, $q|_A = p$ so $\mathrm{RU}(p) \geq \alpha + 1$. □

A direct consequence of (2.2.3) and (2.2.4) is

Corollary 2.2.5. *In* DCF_0*, if* RM = RC*, then* RM = RU.

By a result of Hrushovski and Scanlon [11], we know that RM and RU are in general not equal in DCF_0 (also see Appendix A). By Corollary 2.2.5, we conclude that RM and RC are not always equal in DCF_0 as well.

Finally using the results in this chapter, we argue that the U-rank of a definable set can always be attained.

Proposition 2.2.6. *Let* $X \subset \mathbb{A}^n$ *be a definable over* $\mathcal{K}$*. Then there exists* $\bar{a} \in \mathcal{U}^n$ *such that* $\bar{a} \in X$ *and* $\mathrm{RU}(\bar{a}/\mathcal{K}) = \mathrm{RU}(X/\mathcal{K})$.

Proof. By taking nonforking extension, we can assume $\mathcal{K}$ is an ω-saturated model of DCF_0. The δ-closure, $\overline{X}$, of X has only finitely many components. So without loss of generality, we can assume X is δ-irreducible. Let $\bar{a}$ be the generic point of $\overline{X}$. By quantifier elimination, X is a δ-open subset of $\overline{X}$ so $\bar{a} \in X$. For any $\bar{b} \in \mathcal{U}^n$, let $\mathcal{I}(\bar{b}/\mathcal{K})$ be the δ-ideal

$$\{f \in \mathcal{K}\{y_1, \ldots, y_n\} : f(\bar{b}) = 0\}.$$

Since $\bar{a}$ is the generic point therefore we have $\mathcal{I}(\bar{a}/\mathcal{K}) \subseteq \mathcal{I}(\bar{b}/\mathcal{K})$ for all $\bar{b} \in X$. In turn, we have $\dim_\delta \mathcal{K}\{\bar{b}\} \leq \dim_\delta \mathcal{K}\{\bar{a}\}$. By Lemma 2.1.2 and Proposition 2.2.1, for any $\bar{b} \in X$, we have $\mathrm{RU}(\bar{b}/\mathcal{K}) \leq \omega d + b$ where d is $\mathrm{td}_\delta\,(\mathcal{K}\langle\bar{a}\rangle/\mathcal{K})$. Thus we have,

$$\omega d \leq \mathrm{RU}(\bar{a}/\mathcal{K}) \leq \mathrm{RU}(X/\mathcal{K}) \leq \omega d + b.$$

Clearly if $\mathrm{RU}(X/\mathcal{K})$ is a successor ordinal then the supremum is attained. In the case $\mathrm{RU}(X/\mathcal{K})$ is a limit then the above inequalities show that $\mathrm{RU}(\bar{a}/\mathcal{K}) = \mathrm{RU}(X/\mathcal{K})$. □

In particular, the proof above shows that when X is an algebraic variety then the generic point of X has maximal U-rank.

CHAPTER 3

APPLICATIONS OF THE LASCAR INEQUALITIES

In this section, we assume $\mathcal{K}$ is an ω-saturated differentially closed field.

3.1 Finite Rank Subsets in Affine and Projective Spaces

We identify hypersurfaces of degree d in $\mathbb{P}^n$ with points in $\mathbb{P}^N$, $(N = \binom{n+d}{d} - 1)$, by:

$$\mathbf{a} = [\cdots, a_I, \cdots] \in \mathbb{P}^N \longleftrightarrow H_{\mathbf{a}} = \Big\{ \mathbf{x} \in \mathbb{P}^n : \sum_{|I|=d} a_I x^I = 0 \Big\}.$$

First we have the following observation.

Proposition 3.1.1. *Let X be a definable subset in $\mathbb{P}^n$ of finite U-rank, then any generic hypersurface in $\mathbb{P}^n$ does not meet X.*

Proof. Let $H = \sum a_I x^I$ be a generic hypersurface of degree d in $\mathbb{P}^n$, i.e. $\mathbf{a} = [\cdots, a_I, \cdots]$ is a generic point of $\mathbb{P}^N$. Suppose $\mathbf{x} \in X \cap H$, then using $\mathbf{x}$ we can specify $\mathbf{a}$ in a proper subvariety of $\mathbb{P}^N$ namely the hyperplane defined by $[\cdots, x^I, \cdots]$. Therefore, $\mathrm{RU}(\mathbf{a}/\mathbf{x}) \leq \omega(N-1)$; by the Lascar inequalities we have

$$\mathrm{RU}(\mathbf{a}) \leq \mathrm{RU}(\mathbf{a}, \mathbf{x}) \leq \mathrm{RU}(\mathbf{a}/\mathbf{x}) \oplus \mathrm{RU}(\mathbf{x}) \leq \omega(N-1) \oplus \mathrm{RU}(X) < \omega N.$$

However, this contradicts the fact that $\mathbf{a}$ is a generic point of $\mathbb{P}^N$. Therefore, we conclude that $X \cap H = \emptyset$. $\square$

Before proving the next result, let us consider the following example.

Example 3.1.2. Let $\mathcal{C}$ be the field of constants in $\mathcal{K}$. Consider the following δ-closed subset of the quasiaffine variety $\mathbb{A}^2\setminus\{\bar{0}\}$

$$\mathcal{C}^2\setminus\{\bar{0}\} = \{(x,y) \in \mathbb{A}^2\setminus\{\bar{0}\} : \delta x = \delta y = 0\}.$$

Note that $\mathcal{C}^2\setminus\{\bar{0}\}$ is isomorphic to the δ-closed subset of $\mathbb{A}^3$ given by

$$\delta t_1 = \delta t_2 = 0, \quad (t_1 - at_2)t_3 - 1 = 0$$

where $a \in \mathcal{K}\setminus\mathcal{C}$. However, $\mathbb{A}^2\setminus\{\bar{0}\}$ is not isomorphic to an affine variety. Let us briefly indicate why. Let $U = \mathbb{A}^2 \setminus \{\bar{0}\}$. The zero set of every non-constant polynomial in $\mathcal{K}[x,y]$ is of codimension 1 in $\mathbb{A}^2$, so it cannot be just the origin. From this we see that $\mathcal{O}(U)$, the ring of regular functions on U is $\mathcal{K}[x,y]$. There is a functorial correspondence between morphisms of affine varieties and homomorphisms of rings of regular functions. So if U is affine, then the identity homomorphism of $\mathcal{K}[x,y]$ will induce an isomorphism between U and $\mathbb{A}^2$ but it is easy to check that the identity map actually induces the inclusion from U to $\mathbb{A}^2$. This shows that U is not isomorphic to any affine variety.

Example 3.1.2 is actually a special case of:

Theorem 3.1.3. *Let X be a δ-closed subset of a quasiaffine variety $V \subseteq \mathbb{A}^n$ with $\mathrm{RU}(X) < \omega$. Then X is definably isomorphic to a δ-closed set in $\mathbb{A}^{n+1}$.*

Proof. Let $V = F \cap G$ where F is a Zariski closed set and G is a Zariski open set of $\mathbb{A}^n$. Suppose G is given by $\bigvee_{j=1}^{q} g_j(\bar{t}) \neq 0$, where $g_j \in \mathcal{K}[t_1, \dots, t_n]$, $j = 1, \dots, q$. As a δ-closed subset of V, let X be defined by

$$\bigwedge_{i=1}^{p} f_i(\bar{t}) = 0 \wedge \bigvee_{j=1}^{q} g_j(\bar{t}) \neq 0$$

where $f_i \in \mathcal{K}\{t_1, \dots, t_n\}$, $i = 1, \dots, p$. Let $G_{\bar{z}}(\bar{t}) = \sum_{j=1}^{q} z_j g_j(\bar{t})$ and

$$S = \Big\{(\bar{c}, \bar{x}) \in \mathbb{A}^q \times X : G_{\bar{c}}(\bar{x}) = 0\Big\}.$$

For any $(\bar{c}, \bar{x}) \in S$, since the $g_j(\bar{x})$'s are not all zero, $\bar{c}$ satisfies a nontrivial linear polynomial with coefficients in the δ-field generated by $\bar{x}$ and, B, the canonical base of X. So we have $\mathrm{RU}(\bar{c}/\bar{x}, B) \leq \omega(q-1)$. Thus by the Lascar inequalities, we have

$$\begin{aligned} \mathrm{RU}(\bar{c}/B) &\leq \mathrm{RU}(\bar{c}, \bar{x}/B) \\ &\leq \mathrm{RU}(\bar{c}/\bar{x}, B) \oplus \mathrm{RU}(\bar{x}/B) \\ &\leq \omega(q-1) \oplus \mathrm{RU}(X) < \omega q. \end{aligned}$$

Since $\mathcal{K}$ is ω-saturated, we can choose $\bar{a} \in \mathbb{A}^q$ such that $\mathrm{RU}(\bar{a}/B) = \omega q$. Therefore, for such an $\bar{a}$, we have $(\bar{a}, \bar{x}) \notin S$ for all $\bar{x} \in X$, hence

$$X \subseteq \{\bar{x} \in \mathbb{A}^n : G_{\bar{a}}(\bar{x}) \neq 0\} \subseteq \left\{\bar{x} \in \mathbb{A}^n : \bigvee_{j=1}^{q} g_j(\bar{x}) \neq 0\right\}.$$

Thus X is also given by

$$\bigwedge_{i=1}^{p} f_i(\bar{t}) = 0 \wedge G_{\bar{a}}(\bar{t}) \neq 0.$$

Let $\hat{X}$ be the affine δ-closed set in $\mathbb{A}^{n+1}$ defined by the f_i's and $G_{\bar{a}}(\bar{t})t_{n+1} - 1$. It is easy to see that $\hat{X}$ and X are definably isomorphic via the projection $(a_1, \ldots, a_{n+1}) \to (a_1, \ldots, a_n)$. □

The following is an immediate consequence of Proposition 3.1.1 and Theorem 3.1.3.

Corollary 3.1.4. *Let $V \subseteq \mathbb{P}^n$ be a quasiprojective variety. Let X be a δ-closed subset in V of finite U-rank, then X is definably isomorphic to a δ-closed set in $\mathbb{A}^{n+1}$.*

Proof. By Proposition 3.1.1, $X \subseteq V \setminus H_{\mathbf{a}}$ where **a** is a generic point of $\mathbb{P}^n$. Since $V \setminus H_{\mathbf{a}}$ is quasiaffine, the assertion follows from Theorem 3.1.3 □

By a result of Pillay [30], every finite rank δ-group can be embedded into an algebraic group. Since algebraic groups are quasiprojective, Corollary 3.1.4 yields as a special case the result of Hrushovski and Sokolović's [12] that any finite rank δ-group is differentially affine.

The results we have obtained so far are based on a simple idea—**sets of finite U-rank are "small" so anything related to them cannot possibly be generic**. Along the same line we prove:

Proposition 3.1.5. *Let X be a definable subset of $\mathbb{P}^n$ of finite U-rank, $(n \geq 2)$. Let L be a generic k-plane in $\mathbb{P}^n$ and $\pi_L : \mathbb{P}^n \setminus L \to \mathbb{P}^{n-k-1}$ be a projection from L. Then the restriction of π_L to X is injective.*

Before giving the proof, let us fix our notation. Let $\mathbb{G}(k,n)$ be the set of all k-dimensional linear subspaces (k-planes) of $\mathbb{P}^n$. It is has a projective variety structure. For various $k \leq n$, we call these varieties the Grassmannians.

Let X and Y be two varieties in $\mathbb{P}^n$. Let $\mathcal{J}(X,Y)$ be the closure in $\mathbb{G}(1,n)$ of the locus of lines $\overline{xy}$ with $x \in X$, $y \in Y$ and $x \neq y$. The **join of X and Y**, denoted by $J(X,Y)$, is defined to be the union of all the lines in $\mathcal{J}(X,Y)$. It is a subvariety of $\mathbb{P}^n$.

Proof. Suppose not, then there exist $\mathbf{x}, \mathbf{y} \in X$ such that $J := J(\mathbf{x}, L) = J(\mathbf{y}, L)$. Let U be the Zariski open set $\{H \in \mathbb{G}(k,n) : \mathbf{x} \notin \mathbf{H}\}$. By Proposition 3.1.1, $L \cap X = \emptyset$ so in particular $L \in U$. Let us work over the δ-field generated by $\mathbf{x}$. Consider the following regular map

$$U \xrightarrow{\Phi} \mathbb{G}(k+1,n)$$

$$H \longmapsto J(\mathbf{x}, H)$$

We have $J = \Phi(L)$ which is the generic point of the image $V := \{M \in \mathbb{G}(k+1,n) : \mathbf{x} \in M\}$. So $\mathrm{RU}^-(J) = \omega \dim(V)$. However, using $\mathbf{y}$ we can specify J in the proper subvariety

$$W := \{M \in \mathbb{G}(k+1,n) : \overline{\mathbf{xy}} \subseteq M\}$$

of V. So $\mathrm{RU}(J/\mathbf{y}) \leq \omega(\dim W) < \omega(\dim V - 1)$. Here J is regarded as a point of the Grassmanian instead of a definable set. Since $\mathrm{RU}(\mathbf{y})$ is finite, so by the Lascar inequalities

$\mathrm{RU}(J) \leq \mathrm{RU}(J/\mathbf{y}) \oplus \mathrm{RU}(\mathbf{y}) < \omega(\dim V) = \mathrm{RU}(J)$. This contradiction shows that $\pi_L|_X$ must be injective. □

With this we can prove:

Theorem 3.1.6. *Any finite U-rank subset of $\mathbb{P}^n$ is definably isomorphic to a definable subset of $\mathbb{A}^1$.*

Proof. Let X be a finite U-rank subset of $\mathbb{P}^n$. Pick a generic $(n-2)$-plane, L, in $\mathbb{P}^n$. By Proposition 3.1.5, $\pi_L|_X$ is injective. Its set theoretic inverse, $\mathbf{y} \mapsto J(\mathbf{y}, L) \cap X$, is clearly definable. Therefore $\pi_L X$ is definably isomorphic to X. In particular, $\pi_L X$ also has finite U-rank and hence cannot be the whole $\mathbb{P}^1$. Therefore it is sitting inside a copy of the affine line. □

So as far as those properties invariant under definable isomorphism are concerned, one can reduce the study of finite rank sets to the study of finite rank subsets of the affine line.

We end this chapter with one more application of the Lascar inequalities. This will be useful to us in Section 5.2.

Proposition 3.1.7. *Let X be a proper δ-closed subset of $\mathbb{P}^n$ and $\mathbf{p} \notin X$. Suppose $\mathrm{RU}(X) \geq \omega$. Then $\mathrm{RU}(\pi_{\mathbf{p}}(X)) \geq \omega$.*

Proof. By ω-saturation of $\mathcal{K}$, we can assume both X and $\pi_{\mathbf{p}}$ are definable without using parameters. Let $\mathbf{a} \in \mathbb{P}^n$ be a generic point of X and $\mathbf{b} = \pi_{\mathbf{p}}(\mathbf{a})$. Then we have $\mathbf{a} \in X \cap \overline{\mathbf{pb}}$. The intersection does not contain $\mathbf{p}$ so is a proper δ-closed subset of the projective line $\overline{\mathbf{pb}}$. Hence $\mathrm{RU}(\mathbf{a}/\mathbf{b})$ is finite. By the Lascar inequalities

$$\omega \leq \mathrm{RU}(\mathbf{a}) \leq \mathrm{RU}(\mathbf{a}, \mathbf{b}) \leq \mathrm{RU}(\mathbf{a}/\mathbf{b}) \oplus \mathrm{RU}(\mathbf{b})$$

therefore we have $\omega \leq \mathrm{RU}(\mathbf{b}) \leq \mathrm{RU}(\pi_{\mathbf{p}}(X))$. □

CHAPTER 4

CHAINS OF DIFFERENTIAL SUBVARIETIES IN AN ALGEBRAIC VARIETY

Let X be a δ-variety over a δ-field $\mathcal{K}$. Suppose the δ-transcendental degree of $\mathcal{K}\langle X\rangle$ over $\mathcal{K}$, $\mathrm{td}_\delta(X/\mathcal{K})$, is d. We call an increasing chain of δ-subvarieties of X a **long chain in** X if it has length greater than ωd. In [14], Johnson showed that long chains always exist. The question is: **Given a point $x \in X$, is there a long chain starting at x?** This is called the Kolchin Catenary Problem. We will give an affirmative answer to this question when X is an algebraic variety. Compared to other chapters, we will use more results from algebraic geometry and differential algebra. Our references for algebraic geometry are [29] [1], [8] and [9]. For the definition of prolongation and its basic properties, readers can consult [4] and [15].

4.1 Preliminaries

Let us recall here that our δ-rings contain $\mathbb{Q}$ hence by (1.1.1) every δ-ring has a prime δ-ideal. Also the tensor product of δ-rings has a natural δ-ring structure if it is nonzero.

Our starting point is the following result which asserts that the going-up and the going-down theorems still hold if we replace the set of prime ideals by the set of prime δ-ideals.

Proposition 4.1.1. *Let $\mathcal{A} \subseteq \mathcal{B}$ be δ-integral domains with $\mathcal{B}$ integral over $\mathcal{A}$. Then the lying-over theorem and the going-up theorem hold for the class of prime δ-ideals. Moreover, if $\mathcal{A}$ is integrally closed then the going-down theorem holds for the class of prime δ-ideals as well.*

Proof. Let $\mathfrak{p}$ be a prime δ-ideal in $\mathcal{A}$. By the lying-over theorem, $\mathcal{B} \otimes_{\mathcal{A}} k(\mathfrak{p})$ has a prime ideal so it is not the zero ring. So $\mathcal{S} := \mathcal{B} \otimes_{\mathcal{A}} k(\mathfrak{p})$ is a δ-ring and the canonical map $\mathcal{B} \to \mathcal{S}$ is a δ-homomorphism. By (1.1.1), $\mathcal{S}$ has a prime δ-ideal; its contraction in $\mathcal{B}$ will be a prime δ-ideal lying over $\mathfrak{p}$. This proves the differential version of the lying-over theorem.

The going-up and the going-down theorems for δ-ideals have similar proofs. We will only prove the latter. Suppose $\mathfrak{p}_1 \supsetneq \mathfrak{p}_2$ are prime δ-ideals of $\mathcal{A}$ and $\mathfrak{q}_1$ is a prime δ-ideal of $\mathcal{B}$ lying over $\mathfrak{p}_1$. We have the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{B} & \longrightarrow & \mathcal{B}_{\mathfrak{q}_1} & \longrightarrow & \mathcal{B}_{\mathfrak{q}_1} \otimes k(\overline{\mathfrak{p}_2}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{A} & \longrightarrow & \mathcal{A}_{\mathfrak{p}_1} & \longrightarrow & k(\overline{\mathfrak{p}_2})
\end{array}$$

where $\overline{\mathfrak{p}_2} = \mathfrak{p}_2 \mathcal{A}_{\mathfrak{p}_1}$. Since $\mathcal{A}$ is integrally closed, by the going-down theorem for prime ideals we conclude that there is a prime ideal inside $\mathfrak{q}_1$ lying over $\mathfrak{p}_2$. Therefore $\mathcal{S} := \mathcal{B}_{\mathfrak{q}_1} \otimes_{\mathcal{A}_{\mathfrak{p}_1}} k(\overline{\mathfrak{p}_2})$ is nonzero and is a δ-ring under the natural derivation. So $\mathcal{S}$ has a prime δ-ideal. Let $\mathfrak{q}_2 \subset \mathcal{B}$ be the preimage of a prime δ-ideal of $\mathcal{S}$. It is easy to see that $\mathfrak{q}_2$ is a prime δ-ideal inside $\mathfrak{q}_1$ and lying over $\mathfrak{p}_2$. $\square$

Let us point out that Proposition 4.1.1 is strong enough for our subsequent arguments. However, there is one unsatisfactory point: it only guarantees one of the prime ideals lying over a given prime δ-ideal is closed under the derivation. A natural question would be: **When are they all prime δ-ideals?** We will give a sufficient condition here. First, let us begin with an easy lemma.

Lemma 4.1.2. *Let $(\mathcal{L}, \delta_{\mathcal{L}})$ and $(\mathcal{H}, \delta_{\mathcal{H}})$ be δ-field extensions of $(\mathcal{K}, \delta)$ such that $\mathcal{L}/\mathcal{K}$ is algebraic. Then every $\mathcal{K}$-homomorphism from $\mathcal{L}$ to $\mathcal{H}$ is a δ-homomorphism.*

Proof. Pick any $\varphi \in \mathrm{Hom}_{\mathcal{K}}(\mathcal{L}, \mathcal{H})$ and $x \in \mathcal{L}$. Let f be the minimal polynomial of x over K, then we have

$$0 = \delta_{\mathcal{L}}(0) = \delta_{\mathcal{L}}(f(x)) = f^{\delta}(x) + f'(x)\delta_{\mathcal{L}}(x)$$

where f^{δ} is the polynomial obtained by applying δ to the coefficients of f and f' is the formal derivative of f. Since we are in characteristic 0, x is separable over $\mathcal{K}$ so we have $f'(x) \neq 0$ hence

$$\delta_{\mathcal{L}}(x) = -\frac{f^{\delta}(x)}{f'(x)}.$$

Since φ is a $\mathcal{K}$-homomorphism between differential fields, f is also the minimal polynomial of $\varphi(x)$ over $\mathcal{K}$. By exactly the same argument, we have

$$\varphi(\delta_{\mathcal{L}}(x)) = \varphi\left(-\frac{f^{\delta}(x)}{f'(x)}\right) = -\frac{f^{\delta}(\varphi(x))}{f'(\varphi(x))} = \delta_{\mathcal{H}}(\varphi(x)).$$

This shows that φ is a δ-homomorphism. □

Proposition 4.1.3. *Let $\mathcal{A}$ be an integrally closed δ-domain. Let $\mathcal{K}$ be the field of fractions of $\mathcal{A}$ and $\mathcal{L}$ be a normal algebraic extension of $\mathcal{K}$. Let $\bar{\mathcal{A}}$ be the integral closure of $\mathcal{A}$ in $\mathcal{L}$. Then every prime ideal of $\bar{\mathcal{A}}$ lying over a prime δ-ideal of $\mathcal{A}$ is also a δ-ideal.*

Proof. Let $\mathfrak{p}$ be a prime δ-ideal of $\mathcal{A}$. By Proposition 4.1.1, there is a prime δ-ideal $\mathfrak{q}$ of $\bar{\mathcal{A}}$ lying over $\mathfrak{p}$. Since Char$\mathcal{K} = 0$, $\mathcal{L}/\mathcal{K}$ is a separable extension. So by Theorem 9.3 (iii) in [25], all prime ideals of $\bar{\mathcal{A}}$ lying over $\mathfrak{p}$ are conjugates over $\mathcal{K}$. Now by taking $\mathcal{H} = \mathcal{L}$ in Lemma 4.1.2, we conclude that every $\sigma \in \mathrm{Aut}_{\mathcal{K}}(\mathcal{L})$ is a δ-homomorphism hence $\sigma(\mathfrak{q})$ is a δ-ideal as well. □

The assumptions of the going-down theorem are sufficient to ensure that every prime ideal lying over a prime δ-ideal is again a δ-ideal:

Theorem 4.1.4. *Let $\mathcal{A} \subseteq \mathcal{B}$ be δ-domains. Suppose $\mathcal{A}$ is integrally closed and $\mathcal{B}$ integral over $\mathcal{A}$ then every prime ideal lying over a prime δ-ideal of $\mathcal{A}$ is also a δ-ideal.*

Proof. Let $\mathcal{K} = \langle \mathcal{A} \rangle$ be the field of fractions of $\mathcal{A}$. Since $\mathcal{B}$ is integral over $\mathcal{A}$, $\langle \mathcal{B} \rangle$ is an algebraic extension of $\mathcal{K}$. Let $\mathcal{L}$ be the normal closure of $\langle \mathcal{B} \rangle$ in the algebraic closure of $\mathcal{K}$. Let $\bar{\mathcal{A}}$ be the integral closure of $\mathcal{A}$ in $\mathcal{L}$. Since $\mathcal{B}$ is integral over $\mathcal{A}$, $\bar{\mathcal{A}}$ is also the integral closure of $\mathcal{B}$ in $\mathcal{L}$. Let $\mathfrak{p}$ be any prime δ-ideal of $\mathcal{A}$ and $\mathfrak{q}$ be a prime ideal in $\mathcal{B}$ lying over $\mathfrak{p}$. Since $\bar{\mathcal{A}}$ is integral over $\mathcal{B}$, by the usual lying-over theorem there is a prime $\mathfrak{r}$ lying over $\mathfrak{q}$. In particular, $\mathfrak{r}$ is lying over $\mathfrak{p}$; by Proposition 4.1.3, $\mathfrak{r}$ is a δ-ideal. Thus so is $\mathfrak{q} = \mathcal{B} \cap \mathfrak{r}$. □

In the following, $\mathcal{F}$ will be a differentially closed field of characteristic 0. Let X be a δ-$\mathcal{F}$-variety with $\mathrm{td}_\delta\,(X/\mathcal{F}) = d$. We will show that if X is an algebraic variety then there is a long chain starting at any point of X.

Remark 4.1.5. Let us point out a few facts before restricting ourselves to the case where X is an algebraic variety.

1. The existence of long chain is a local problem: Let x be any point in X and U be a δ-open neighborhood of x. If we can find a long chain

$$x = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_{\omega d}$$

in U, then the δ-closure of the Y_α's in X will form a long chain in X. So when X is an $\mathcal{F}$-variety we can assume X is affine.

2. In fact one can go the other way: Suppose $Z_0 \subsetneq Z_1$ are two δ-subvarieties of X and U is a δ-open subset of X such that $U \cap Z_0$ is nonempty. Since the Z_i's are δ-irreducible, they are equal to the δ-closure of their intersections with U. So we must have $U \cap Z_0 \subsetneq U \cap Z_1$. Therefore, if $\boldsymbol{C} = \{C_i\}$ is any infinite increasing chain of δ-subvarieties in X, then the chain obtained by intersecting $\boldsymbol{C}$ with U will have the same length as $\boldsymbol{C}$ provided that $C_i \cap U$ is nonempty for some $i < \omega$.

3. The Kolchin Catenary Problem was solved "generically": For any δ-$\mathcal{F}$-variety X, there is a proper δ-closed subset Y of X such that for any $x \in X \setminus Y$ there is a long chain starting at x. One can see this by going through Johnson's arguments in [14]. In fact, before Johnson's paper, Rosenfeld gave a necessary (δ-closed) condition which guarantee the existence of long chain (cf. [31]). Moreover he dealt with chains starting from δ-subvarieties of arbitrary δ-transcendence degree instead of just points.

So the essence of Kolchin's problem is: what happens at the singular points? Our belief is: By resolution of singularities in characteristic 0 for algebraic varieties, one should be able to answer the question in that case.

4.2 The Étale Case

We know that a long chain exists at every point of $\mathbb{A}^d$. For example, the chain of prime δ-ideals

$$[y_1, y_2, \ldots, y_d] \supset [\delta y_1, y_2, \ldots, y_d] \supset [\delta^2 y_1, y_2, \ldots, y_d] \supset \\ \cdots \supset [y_2, \ldots, y_d] \supset [\delta y_2, \ldots, y_d] \supset \cdots \supset [0]$$

of $\mathcal{F}\{y_1, \ldots, y_n\}$ corresponds to a long chain $\boldsymbol{S}$ starting at the origin. In fact, we know a little bit more:

Proposition 4.2.1. *Let U be a Zariski open subset of $\mathbb{A}^d$. Then there is a long chain in $\mathbb{A}^d$ starting from the origin such that its intersection with U is still a long chain in U.*

Proof. First note that $S_1 = \{\bar{x} \in \mathbb{A}^d : \delta x_1 = x_2 = \cdots = x_d = 0\}$ is the set of constant points on the y_1-axis. By an automorphism g of $\mathbb{A}^d$, we can make sure $g(U)$ intersects the y_1-axis. Therefore, the complement of $g(U)$ in the y_1-axis is a finite set and hence cannot be all S_1. That means $S_1 \cap g(U) \neq \emptyset$, hence $g^{-1}(\boldsymbol{S})$ will be such a chain by Remark 4.1.5 (2). Certainly if $\bar{0} \in U$, then $g^{-1}(\boldsymbol{S}) \cap U$ will be a long chain in U starting from the origin. $\square$

Let X be an affine $\mathcal{F}$-variety of dimension d; by Noether Normalization, there is a finite surjection from X to $\mathbb{A}^d$. The idea is to get a long chain at x by applying the differential going-down theorem. On the ring level, let $A = \mathcal{F}[y_1, \dots, y_d]$ and $B = \mathcal{F}[X]$, we have an embedding of A into B such that B is a finite A-module. In the light of Proposition 4.1.1, the above argument will present no problems as long as the property of "being finite" can be infinitely prolonged. That is, if $\mathcal{F}\{X\}$ is a finite $\mathcal{F}\{y_1, \dots, y_d\}$ module. However, we may run into trouble even at the very first step. For example, take B to be the coordinate ring of the plane curve $y^2 = x^3 + 1$ and $A = \mathcal{F}[x]$. It is easy to see that the image of δy in the first prolongation, $B_1 = \mathcal{F}[x, \delta x, y, \delta y]/(y^2 - x^3 + 1, 2y\delta y - 3x^2\delta x)$, is not integral over $A_1 = \mathcal{F}[x, \delta x]$. Hence B_1 is not a finite A_1-module. However, in this example, the problem happens exactly at the branch points of the corresponding surjection. In fact, one can easily check that δy is integral over x and δx after localizing to the open set $\{x \in \mathbb{A}^1 : x^3 - 1 \neq 0\}$.

We will start with the case where there are no branch points and make the argument above precise.

Definition 4.2.2. Let $f : X \to Y$ be a morphism of varieties. Let $x \in X$ and $f(x) = y$. We say that f is **unramified at** x if $\mathfrak{m}_x = \mathfrak{m}_y \cdot \mathcal{O}_{x,X}$, and $k(x)$ is a separable finite extension of $k(y)$. This is also equivalent to the surjectivity of the canonical map $f^*\Omega^1_{Y/\mathcal{F}} \to \Omega^1_{X/\mathcal{F}}$ at x (see Chapter VI Proposition 3.6 in [1]). We say f is **unramified** if f is unramified at every $x \in X$.

Since we are in the characteristic 0 case, the field extension $k(x)/k(y)$ will always be separable. Moreover, if $x \in X$ is a closed point, since $\mathcal{F}$ is algebraically closed we have $k(x) = k(y) = \mathcal{F}$. So x is an unramified point of f if $\mathfrak{m}_x = \mathfrak{m}_y \cdot \mathcal{O}_x$.

We say that a morphism is **étale** if it is both flat and unramified. It is an **étale covering** if it is both étale and surjective.

We will use the following result in [29] (Chapter 3, §5, Theorem 4) to show that a map is étale instead of verifying the definition directly.

Theorem 4.2.3. *Let $f : X \to Y$ be a morphism of nonsingular n-dimensional varieties over an algebraically closed field k. If for all closed points $y \in Y$, the fibre $f^{-1}(y)$ is a finite set of reduced points (or, equivalently, if for all closed points $x \in X$, $\mathfrak{m}_x = f^*(\mathfrak{m}_{f(x)})\mathcal{O}_x$), then f is étale.*

The following result of Buium ([4] Chapter 4, Prop. 4.6) describes the canonical prolongation of an étale covering between smooth $\mathcal{F}$-varieties.

Proposition 4.2.4 (Buium). *Let $f : X \to Y$ be an étale finite covering of smooth $\mathcal{F}$-varieties and let $f^r : X^r \to Y^r$ be the induced morphisms between the canonical infinite prolongation sequences. Then the squares*

$$\begin{array}{ccc} X & \longleftarrow & X^r \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^r} \\ Y & \longleftarrow & Y^r \end{array}$$

are Cartesian. Consequently if f is a Galois covering with group G then

$$\mathcal{O}^{[r]}(Y) = \mathcal{O}^{[r]}(X)^G, \quad r \geq 0.$$

With this result we can prove:

Corollary 4.2.5. [2] *Let $f : X \to Y$ be an étale finite covering of smooth affine varieties, then $X^\infty \simeq X \times_Y Y^\infty$. Therefore the induced map $f^\infty : X^\infty \to Y^\infty$ is still a finite étale covering.*

Proof. Since finite, étale and surjectivity are properties preserved under base change, all we need to prove is that the diagram

$$\begin{array}{ccc} X & \longleftarrow & X^\infty \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^\infty} \\ Y & \longleftarrow & Y^\infty \end{array}$$

is Cartesian. Let Z be the projective limit of the system $\{X \times_Y Y^r : r \geq 0\}$. By the universal property of Y^∞ (as a projective limit), there exists a unique morphism π from Z to Y^∞ making the following diagram commutes.

$$\begin{array}{ccc} X & \longleftarrow & Z \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \longleftarrow & Y^\infty \end{array}$$

By easy diagram chasing, one checks that the above diagram is Cartesian. Hence Z is isomorphic to $X \times_Y Y^\infty$. Finally by Proposition 4.2.4, the projective system

$$X \leftarrow X \times_Y Y^1 \leftarrow \cdots \leftarrow X \times_Y Y^r \leftarrow \cdots$$

[2]The same statement in terms of ring and Hasse-Schmidt differential appears as Lemma 6 in [7].

is isomorphic to the canonical infinite prolongation sequence of $X \to \operatorname{Spec}\mathcal{F}$. Therefore $Z \simeq X^\infty$ by uniqueness of projective limit. □

We need one more observation before proving the existence of long chain in the étale case.

Lemma 4.2.6. *Let $X = \operatorname{Spec} A$ be a smooth affine $\mathcal{F}$-variety. Then, A_∞, is integrally closed.*

Proof. Let $X^r = \operatorname{Spec} A_r$ be the r-th term in the canonical infinite prolongation of $X \to \operatorname{Spec}\mathcal{F}$. By [4] Chapter 3 (2.5), all the X^r are smooth affine varieties. In particular, the A_r's are integrally closed. Also in the smooth case, the map $X^{r+1} \to X^r$ has a natural structure of a torsor under the relative tangent bundle $\boldsymbol{V}(T_{X^r/X^{r-1}}) \to X^r$. In particular, $X^{r+1} \to X^r$ is surjective and hence the induce map $A_r \to A_{r+1}$ is injective. Let K_r be the field of fractions of A_r. Identifying A_r as a subring of A_{r+1}, we have $A_\infty \cong \bigcup_r A_r$. This also induces an isomorphism between K_∞, the field of fractions of $\mathcal{A}_\infty$, and $\bigcup K_r$. Now A_∞ is integrally closed follows from the fact that each A_r is integrally closed. □

For later applications, only a very special case of Lemma 4.2.6 is needed, namely the case when $A = \mathcal{F}[y_1, \dots, y_n]$. In fact, in this case $A_\infty = \mathcal{F}\{y_1, \dots, y_n\}$ is a polynomial ring, so in particular it is integrally closed.

Proposition 4.2.7. *Let $f : X \to Y$ be an étale finite covering of smooth $\mathcal{F}$-varieties. Suppose $W_0 \subset W_1 \subset \cdots \subset W_\alpha$ is an increasing chain of δ-subvarieties of Y and V_0 is a δ-subvariety of X such that $f(V_0) = W_0$ then there exists an increasing chain*

$$V_0 \subset V_1 \subset \cdots \subset V_\alpha$$

of δ-subvarieties of X such that $f(V_\beta) = W_\beta$ for every $\beta \le \alpha$.

Proof. By Remark 4.1.5, we can assume X and Y are affine. Say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let $\mathfrak{q}_0$ and $\mathfrak{p}_\beta$ ($\beta \le \alpha$) be prime δ-ideals in B_∞ and A_∞ respectively corresponding to V_0 and W_β. Since $f(V_0) = W_0$, we have $\mathfrak{q}_0$ lying over $\mathfrak{p}_0$. By Corollary 4.2.5, $\mathcal{A}_\infty \subset \mathcal{B}_\infty$ and B_∞ is a finite A_∞ module so in particular B_∞ is integral over A_∞. We also have A_∞ is integrally closed by Lemma 4.2.6. Now we can apply the differential going-down theorem (4.1.1) inductively to build a decreasing chain of prime δ-ideals lying over $\{\mathfrak{p}_\beta\}_{\beta \le \alpha}$. At stage $\beta + 1$, we have $\mathfrak{q}_\beta$ lying over $\mathfrak{p}_\beta \supset \mathfrak{p}_{\beta+1}$. So by (4.1.1), we get $\mathfrak{q}_{\beta+1}$. At stage γ where γ is a limit, we apply (4.1.1) to $\bigcap_{\beta<\gamma} \mathfrak{p}_\beta \supseteq \mathfrak{p}_\gamma$ and get $\mathfrak{q}_\gamma \subseteq \bigcap_{\beta<\gamma} \mathfrak{q}_\beta$ such that $\mathfrak{q}_\gamma \cap \mathcal{A}_\infty = \mathfrak{p}_\gamma$. By our construction, the δ-subvarieties V_β's correspond to the $\mathfrak{q}_\beta$'s form an increasing chain with $f(V_\beta) = W_\beta$ for all $\beta \le \alpha$. □

4.3 The Smooth Case

In the next two sections we identify $\mathcal{F}$-varieties with their $\mathcal{F}$-points. Our next goal is to prove the following statement:

> *Let X be a smooth affine $\mathcal{F}$-variety of dimension d. Then for any $x \in X$, there exists a finite surjection $\psi : X \to \mathbb{A}^d$ which is étale on some neighborhood of x.*

In fact, we will prove something slightly stronger (see Proposition 4.3.3). Once we establish this, by going to a smaller affine neighborhood of $\psi(x)$, we reduce ourselves to the situation described in Proposition 4.2.7.

The next two propositions are auxiliary. The first one is a minor variation of a result in [29].

Proposition 4.3.1. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension d and x be a smooth point of X. Suppose H is a hyperplane in $\mathbb{P}^n$ and $x \notin H$. Then there exists a linear subspace, M, of H such that the projection $\pi : \mathbb{P}^n \setminus M \to \mathbb{P}^{d+1}$ is an isomorphism near x.*

Proof. Suppose $n > d+1$ otherwise the statement is obvious. Following the arguments in [29] (Chapter 3 §7. Theorem 2), it suffices to find a linear subspace M of H such that:

1. $J(x, M) \cap X = \{x\}$ where $J(x, M)$ is the join of x and M.

2. $T_{x,X} \cap M = \emptyset$ where $T_{x,X}$ is the projective tangent space of X at x.

The fact that x is a smooth point implies $J(x, X)$ contains $T_{x,X}$ and $\dim J(x, X) = \dim(X) + 1 = d + 1$. Since $x \notin H$, $J(x, X) \cap H$ is a proper Zariski closed subset of $J(x, X)$ hence can have dimension at most d (in fact it has dimension d). Therefore one can find a linear subspace M of H of dimension $n - d - 2$ such that M does not meet $J(x, X)$. This M will satisfy both condition (1) and (2). ☐

Proposition 4.3.2. *Let X be a codimension 1 subvariety of $\mathbb{P}^{d+1}$. Suppose $x \in X$ is a smooth point and H is a hyperplane not containing x. Then there exists a projection $\varphi : \mathbb{P}^{d+1} \dashrightarrow \mathbb{P}^d$ with center at some point $p \in H$ such that $\varphi|_X$ is a finite surjection and it is étale over some affine neighborhood of $\varphi(x)$.*

Proof. Let $Bl_x(X)$ be the blow-up of X at x. Consider the map $\tilde{\pi}_x : Bl_x(X) \to H$ induced by the projection π_x which sends $w \in X \setminus \{x\}$ to $\overline{xw} \cap H$. The branch locus of $\tilde{\pi}_x$ is a closed subset of codimension at least 1 in H. Since $X \cap H$ is a proper closed subset of X, by counting dimensions, we have $d-1 \geq \dim(X \cap H) \geq d+d-(d+1) = d-1$. Therefore $\dim(X \cap H) = d-1$. Since x is a smooth point, T_x, the projective tangent space of X at x is d-dimensional. By a similar argument, $\dim(T_x \cap H) = d-1$. Thus we can find a point $p \in H$ avoiding the branch locus of $\tilde{\pi}_x$, $X \cap H$ and $T_x \cap H$ all together. Let φ be a projection to $\mathbb{P}^d$ with center at p. We argue that φ will satisfy the requirements.

Since $p \notin X$, by an argument in [29] (Chapter 2, §7, proposition 6), $\varphi|_X : X \to \mathbb{P}^d$ is a finite morphism. Moreover, X has codimension one, so the map is surjective. For simplicity, we will write φ instead of $\varphi|_X$. It remains to show that φ is étale over a neighborhood of $y = \varphi(x)$. First let us show that φ is unramified at every point of the fiber $\varphi^{-1}(y)$. The necessary and sufficient condition for $z \in X$ to be an unramified point of φ is $p \notin T_z$. Since we choose p away from T_x, this guarantee φ is unramified at x. Now if w is another point in the fiber, we have p, x and w collinear. So

$$w \text{ is a ramified point of } \varphi \iff p \in T_w \iff \overline{xw} = \overline{xwp} = \overline{wp} \subseteq T_w.$$

Note that the condition $\overline{xw} \subseteq T_w$ is also equivalent to that w is a ramification point of $\tilde{\pi}_x$. But then p will be a branch point of $\tilde{\pi}_x$. This contradicts our choice of p. Now it follows that none of the points in the fiber over y ramify. In fact, they are nonsingular points: indeed if $z \in \varphi^{-1}(y)$ is a singular point then $\dim \Omega_{X,z} = d+1$ so the map $\varphi^*\Omega_{\mathbb{P}^d} \to \Omega_X$ could not be surjective at z which contradicts the fact that φ is unramified at z. Since φ is finite and $\mathbb{P}^d$ is normal, an application of the going-down theorem shows that φ is an open map (cf. [25] Theorem 9.6). So by going to a smaller affine neighborhood of y, we can assume φ is an unramified finite surjective map between nonsingular varieties. In particular, $\mathfrak{m}_z = \varphi^*(\mathfrak{m}_{\varphi(z)})\mathcal{O}_z$ for all z in the domain. By Theorem 4.2.3, we have that φ is an étale map. □

Combining both (4.3.1) and (4.3.2) we get:

Proposition 4.3.3. *Let X be a smooth affine variety of dimension d. For any $x \in X$, there exists $\psi : X \to \mathbb{A}^d$ which is a finite surjection and is étale over some affine neighborhood of $\psi(x)$.*

Proof. Embed X into $\mathbb{P}^n_{\mathbf{x}}$ (for some $n \geq d+1$) via $X \subset \mathbb{A}^n = \mathbb{P}^n \setminus H$ where $H = \{x_0 = 0\}$. Let $\bar{X}$ be the Zariski closure of X in $\mathbb{P}^n$. Applying Proposition 4.3.1 to $\bar{X}$ and H, we get a projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ centered at M, a linear subspace of H, and a neighborhood W of x such that $\pi|_W$ is an isomorphism between W and $V := \pi(W)$. Note that $\pi(H \setminus M)$ is a hyperplane in $\mathbb{P}^{d+1}$ which does not contain $\pi(x)$ and $\pi(\bar{X})$ is of codimension 1 in $\mathbb{P}^{d+1}_{\mathbf{y}}$. So by (4.3.2), there exists some $p \in \pi(H \setminus M)$ and projection φ at p such that $\varphi|_{\pi(\bar{X})}$ is a finite surjection which is étale over some neighborhood U of $\varphi(\pi(x))$. Since $p \in \pi(H \setminus M)$, the composition $\varphi \circ \pi$ is undefined on a $n - d - 1$ dimensional subspace of H, say L. And it is easy to see the $\varphi \circ \pi|_{\mathbb{P}^n \setminus L}$ is a projection from L, call it π_L. By an automorphism of $\mathbb{P}^d_{\mathbf{z}}$, if necessary, we can assume π_L is given by the linear forms $l_0 = x_0, l_1, \ldots, l_d$. In this case, if we set $V_0 = \mathbb{P}^d \setminus \{z_0 = 0\}$ then $\pi_L^{-1}(V_0) \cap \bar{X} = \bar{X} \setminus H = X$. Therefore $\psi := \pi_L|_X$ is a finite map from X onto $V_0 \cong \mathbb{A}^d$. Since étale maps are open, we see that ψ is étale over some open neighborhood of $\psi(x)$ (e.g. one can take $\varphi(\varphi^{-1}(U \cap V_0) \cap V))$. ☐

The next result settles the question when X is a smooth variety.

Theorem 4.3.4. *Let X be a smooth $\mathcal{F}$-variety of dimension d. Then for any $x \in X$ there is a long chain starting at x.*

Proof. As we have pointed out in Remark 4.1.5, one can assume X is affine to begin with. By Proposition 4.3.3, there is a finite surjection $f : X \to \mathbb{A}^d$ which is étale over an affine neighborhood U of $f(x)$ which we can assume to be 0. Therefore we have the following situation:

$$\begin{array}{ccc} x \in V & \longrightarrow & X \\ \downarrow{\scriptstyle f|_V} & & \downarrow \\ 0 \in U & \longrightarrow & \mathbb{A}^d \end{array}$$

where $V := f^{-1}(U)$. Since étale maps are affine, so V is affine and hence $f|_V$ is an étale finite covering between smooth affine varieties. By Proposition 4.2.1, we get a long chain in U starting at 0. Now the theorem follows from Proposition 4.2.7. ☐

4.4 The General Case

In order to prove the existence of long chain without the smoothness assumption, we need to sharpen Theorem 4.3.4 slightly.

Proposition 4.4.1. *Let X be a smooth $\mathcal{F}$-variety of dimension d, $x \in X$ and E be a codimension one Zariski closed subset of X containing x. Then there exists a long chain $\{W_\alpha\}$ in X starting at x such that $W_\beta \setminus E \subsetneq W_\gamma \setminus E$, for all $\beta < \gamma$.*

Proof. We keep the same notation and assumptions as in Theorem 4.3.4. Let H be a codimension 1 closed subset of $\mathbb{A}^d$ containing $f(E)$. Applying (4.2.1) to the open set $U \setminus H$, we get a long chain $\{Z_\alpha\}$ in $\mathbb{A}^d$ starting at 0 such that $Z_\beta \cap (U \setminus H) \subsetneq Z_\gamma \cap (U \setminus H)$, for all $\beta < \gamma$. By Proposition 4.2.7, there is a long chain $\boldsymbol{C}$ in V lying over $\{Z_\alpha \cap U\}$. The chain, $\{W_\alpha\}$, obtained by taking δ-closure in X of each term of $\boldsymbol{C}$ will have the require properties. □

Finally, we use resolution of singularities to prove the general case.

Theorem 4.4.2. *The smoothness assumption in Theorem 4.3.4 is unnecessary.*

Proof. As before, we may well assume X is affine and x is contained in every singular component of X. Let Z be the union of all the singular components. By resolution of singularities in characteristic 0, we have a smooth model $\pi : \tilde{X} \to X$. Let $E = \pi^{-1}(Z)$ be the exceptional divisor. Pick any $\tilde{x} \in \pi^{-1}(x)$, by (4.4.1) we get a long chain $\{W_\alpha\}$ starting at $\tilde{x}$ such that for all $\beta < \gamma$, $W_\beta \setminus E \subsetneq W_\gamma \setminus E$. Let Y_α be the δ-closure of $\pi(W_\alpha)$ in X. Since π is an isomorphism away from E, $\pi(W_\alpha \setminus E) = \pi(W_\alpha) \setminus Z$ is a δ-subvariety of (in particular δ-closed in) $X \setminus Z$. Therefore $\pi(W_\alpha) \setminus Z = Y_\alpha \setminus Z$. So the Y_α's form a long chain in X starting at x. □

CHAPTER 5

COMPLETE SETS IN DIFFERENTIALLY CLOSED FIELDS

Completeness is one of the most fundamental notion in algebraic geometry. In this chapter, we will examine the analogous concept in the case of differential algebraic geometry. The methods that we use here come from both model theory and differential algebra. The model theoretic part is taken from a paper by van den Dries [35]. Using what he called a "Lyndon-Robinson type" result, van den Dries gave a proof of the main theorem of classical elimination theory. The completeness of projective varieties easily follows from this.

Differential completeness was studied by Kolchin in [18]. In that paper, he proved that the set of constant points of a projective space is differentially complete while the whole projective space is not. However, those were the only examples and differentially complete sets have not been extensively studied since then.

Throughout this chapter, $\mathcal{F}$ is a fixed differentially closed field of characteristic 0 and $\mathcal{C}$ is the field of constants of $\mathcal{F}$. All δ-varieties in this chapter are defined over $\mathcal{F}$. We identify a δ-variety X with $X(\mathcal{F})$, the set of $\mathcal{F}$-points of X. The language for our discussion is $\mathfrak{L}_\delta(\mathcal{F}) = \mathfrak{L}_\delta \cup \{c_f : f \in \mathcal{F}\}$ where c_f is a constant symbol for the element $f \in \mathcal{F}$. In this context, we show that every δ-complete set is affine and is definably isomorphic to a δ-complete subset of the line $\mathbb{A}^1$. This is exactly "opposite" to the phenomenon that happens in algebraic geometry since a complete variety is never affine unless it is a point. Next we obtain a criterion for δ-completeness. Using this criterion, we find a new family of δ-complete sets. Some of them are orthogonal to the field of constants. Hence, from the model theoretic point of view, these δ-complete sets are quite different from the set of constant points in a projective space.

5.1 Differential Complete Sets and their Properties

The following notion is the differential counterpart of completeness in algebraic geometry.

Definition 5.1.1. A qusaiprojective δ-variety X is **δ-complete** if the second projection map $\pi_2 : X \times Y \to Y$ is a δ-closed map for every quasiprojective δ-variety Y.

We record in the the next proposition a few properties of δ-complete sets that we are going to use later on.

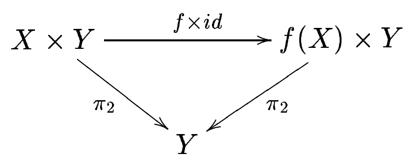
Proposition 5.1.2. *Let X be a δ-complete set and Y a quasiprojective δ-variety.*

1. *Let $f : X \to Y$ be continuous in the δ-topology. Then $f(X)$ is δ-closed in Y and is δ-complete.*

2. *Any δ-closed subset of X is δ-complete.*

3. *If Y is another δ-complete set then so is $X \times Y$.*

Proof. (1) Let Y be a subset of $\mathbb{P}^m$. Then f is a map from X to $\mathbb{P}^m$ which factors through Y. If $f(X)$ is δ-closed in $\mathbb{P}^m$, it is δ-closed in Y. Therefore by replacing Y by $\mathbb{P}^m$, we consider the map $f \times id : X \times \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^m$. The graph of f is δ-closed since it is the inverse image of the diagonal in $\mathbb{P}^m \times \mathbb{P}^m$. By the δ-completeness of X so is $f(X) = \pi_2($ graph of $f)$. Moreover, $f(X)$ is δ-complete since the following diagram commutes:



(2) For any δ-closed $Z \subset X$, the natural inclusion $j: Z \times Y \to X \times Y$ is a δ-closed map. Since the projection $Z \times Y \to Y$ is j followed by $X \times Y \to Y$, the assertion holds.

(3) The projection $X \times Y \times Z \to Z$ factors through $Y \times Z \to Z$ $\square$

Remark 5.1.3. Suppose $X \subseteq \mathbb{P}^n$ is δ-complete. By choosing f to be the canonical embedding in (5.1.2 (1)), we see that X has to be δ-closed in $\mathbb{P}^n$. And we will assume so when we want to show that X is δ-complete. Also, it is not restrictive to require X to be δ-irreducible in the definition; as it is easy to see that the projection $X \times Y \to Y$ is δ-closed for all quasiprojective δ-variety Y if and only if every δ-irreducible component of X is δ-complete. Moreover note that whether a subset is δ-closed is a local question, so by a standard reduction argument (cf [33] Ch.1, 5, Thm. 3), **"X is δ-complete" is equivalent to: for every $m \in \mathbb{N}$, the second projection map $\pi_2: X \times \mathbb{A}^m \to \mathbb{A}^m$ is a δ-closed map**.

5.2 Differentially Complete Sets are Affine

There is a fundamental difference between δ-completeness and completeness in algebraic geometry: **projective spaces are complete but not δ-complete**. In [18], Kolchin gives a family of examples which shows that $\mathbb{P}^n$ is not δ-complete for every $n \geq 1$. Here we will give Kolchin's example for $\mathbb{P}^1$ explicitly by exhibiting the defining equations of a δ-closed set in $\mathbb{P}^1 \times \mathbb{A}^1$ and argue that its projection to $\mathbb{A}^1$ is not δ-closed.

Proposition 5.2.1. *$\mathbb{P}^n$ is not δ-complete.*

Proof. Let us recall the δ-closed set, Z, in Example 1.2.1 is defined by

$$z\delta(y)^2 + (y^4 - 1) = 0$$

$$2z\delta^2(y) + \delta(z)\delta(y) + 4y^3 = 0.$$

We have already shown that Z is a δ-closed set in $\mathbb{P}^1 \times \mathbb{A}^1$. Let $b \in \mathcal{U}$ be δ-transcendental over $\mathcal{F}$. Since $\mathcal{U}$ is differentially closed, there exists $a \in \mathcal{U}$ solution of $b\delta(y)^2 + (y^4 - 1) = 0 \wedge y^4 - 1 \neq 0$. In particular δa cannot be 0. Therefore (a, b) is a point on Z and hence $b \in \pi_2 Z$. So if $\pi_2 Z$ were δ-closed, it has to be the whole $\mathbb{A}^1$. However, since $y^4 - 1$ and $2y^3$ have no common zeros, thus $\pi_2 Z$ does not contain 0. From this we conclude that $\pi_2 Z$ cannot be δ-closed. This shows that $\mathbb{P}^1$ is not δ-complete. Finally since $\mathbb{P}^{n-1}$ can be embedded as a δ-closed subset of $\mathbb{P}^n$, we conclude that $\mathbb{P}^n$ is not δ-complete by Proposition 5.1.2 (2). $\square$

Using the fact that $\mathbb{P}^n$ is not δ-complete, we can argue that δ-complete sets are "small".

Theorem 5.2.2. *Every δ-complete set is of finite U-rank.*

Proof. Suppose X is an infinite U-rank δ-complete set in $\mathbb{P}^n$. By Proposition 3.1.7, the projections of X will also have infinite U-rank. Note that $\mathbb{P}^1$ is irreducible in the δ-topology and $\mathrm{RU}(\mathbb{P}^1) = \omega$, therefore any infinite U-rank δ-closed subset of $\mathbb{P}^1$ must be the whole space. So by repeatedly projecting X to lower dimensional spaces, we conclude that the image must be the whole projective space which is not δ-complete by Proposition 5.2.1. However, this contradicts Proposition 5.1.2 (1). $\square$

We can now show that every δ-complete set is affine.

Corollary 5.2.3. *Every δ-complete set is affine and is definably isomorphic to a δ-complete set in $\mathbb{A}^1$.*

Proof. Let X be δ-complete. By Theorem 5.2.2, $\mathrm{RU}(X) < \omega$. Let $\mathcal{K}$ be an ω-saturated extension of $\mathcal{F}$. Then by Corollary 3.1.4 and Theorem 3.1.6, we know

> $\mathcal{K} \models$ "X is in the complement of a hyperplane" $\wedge$ "there exists a definable isomorphism from X into $\mathbb{A}^1$".

So by model completeness of DCF_0, $\mathcal{F}$ witnesses the same statement. Finally, since the definable isomorphism is a composition of projections, in particular it is continuous with respect to the δ-topology, therefore by Proposition 5.1.2 (1) the image is a δ-complete set also. $\square$

5.3 A Model Theoretic Fact

Using results from the previous section, we can assume that X is an affine δ-closed set when we want to show that X is δ-complete. Our next goal is to derive a general test to decide if a given affine δ-closed set is δ-complete. Our approach to this problem is quite näive, namely verify the definition directly. For each $m \in \mathbb{N}$, we pick an arbitrary δ-closed subset of $X \times \mathbb{A}^m$ and argue that its projection is δ-closed. The reason why model theoretic methods are useful here is that the basic geometric objects that we are interested in, affine δ-closed sets in this case, are definable. In fact, an affine δ-closed set is nothing more than a positive quantifier free definable set in $\mathfrak{L}_\delta(\mathcal{F})$. With this observation, we see that the first thing we need is a way to tell whether a given formula is equivalent to a positive quantifier free formula. Fortunately, in [35] van den Dries told us one such test.

Proposition 5.3.1 (van den Dries). *Let $\mathfrak{L}$ be a first order language. Let T be a complete $\mathfrak{L}$-theory and $\phi(v_1, \ldots, v_m)$ an $\mathfrak{L}$-formula* [3] *(without parameters), then the following are equivalent:*

1. *There exists a positive quantifier free formula ψ such that*

$$T \vdash \forall \bar{v} \; \phi(\bar{v}) \longleftrightarrow \psi(\bar{v}).$$

2. *For any models K and L of T and each homomorphism $f : A \to L$ from a substructure A of K into L we have:*

$$\text{if } \bar{a} \in A^m \text{ and } K \models \phi(\bar{a}), \text{ then } L \models \phi(f(\bar{a})).$$

---

[3] In case $m = 0$, we assume that $\mathfrak{L}$ has a constant symbol.

Proof. 1) $\Rightarrow$ 2) Follows from the definition of homomorphism between $\mathfrak{L}$-structures.

2) $\Rightarrow$ 1) Consider the set of formulas,

$$\Gamma = \{\gamma(\bar{v}) : \gamma \text{ is a positive quantifier free formula and}$$
$$T \vdash \forall \bar{v}\ \gamma(\bar{v}) \to \phi(\bar{v})\}.$$

If for some $\gamma_1, \ldots, \gamma_k \in \Gamma$, $T \vdash \forall \bar{v}\ \phi(\bar{v}) \to \gamma_1(\bar{v}) \vee \cdots \vee \gamma_k(\bar{v})$ then we are done. Aiming at a contradiction, suppose $T^* := T \cup \phi(\bar{c}) \cup \{\neg\gamma(\bar{c}) : \gamma \in \Gamma\}$ is consistent, where $\bar{c}$ are new constant symbols. Let $(K, \bar{a})$ be a model of T^*. Let A be the substructure of K generated by $\bar{a}$. Denote by $\mathrm{Diag}^+(\bar{a})$ the positive diagram of $\bar{a}$ in K, which is the set $\{\gamma(\bar{c}) : \gamma(\bar{v}) \text{ is an atomic formula and } K \models \gamma(\bar{a})\}$. Note that for any $L \models T \cup \mathrm{Diag}^+(\bar{a})$, there is a homomorphism f from A into L. It is obtained by extending the map $\bar{a} \mapsto \bar{c}^L$ in the obvious way. Since $K \models \phi(\bar{a})$, so by our assumption $L \models \phi(f(\bar{a}))$. Now since L is an arbitrary model of $T \cup Diag^+(\bar{a})$, we conclude by compactness that there is $\gamma := \bigwedge \gamma_i$, a conjunction of formulas in $\mathrm{Diag}^+(\bar{a})$ such that $T \vdash \forall \bar{v}\ \gamma(\bar{v}) \to \phi(\bar{v})$. So on one hand $\gamma \in \Gamma$ and $K \models T^*$, therefore $K \models \neg\gamma(\bar{a})$, but on the other hand $K \models \gamma(\bar{a})$ since $\gamma_i \in \mathrm{Diag}^+(\bar{a})$ for each i. This is a contradiction. □

5.4 Valuative Criterion for Differential Completeness

We continue our search for a test for δ-completeness. The result of van den Dries in the previous section ties this problem to the problem of extending δ-homomorphisms. The latter subject had been studied by P.Blum in [3] and S.Morrsion in [27] and [28]. Here we gather some basic facts and introduce some terminology that we are going to use.

Let $\mathcal{K}$ be a δ-field, consider the set

$$\mathbf{H}_{\mathcal{K}} = \{(\mathcal{A}, f, \mathcal{L}) : \mathcal{A} \text{ is a } \delta\text{-subring of } \mathcal{K},\ \mathcal{L} \text{ is a } \delta\text{-field and}$$
$$f : \mathcal{A} \to \mathcal{L} \text{ is a } \delta\text{-homomorphism.}\}$$

Let $(\mathcal{A}_i, f_i, \mathcal{L}_i) \in \mathbf{H}_{\mathcal{K}},\ i = 1, 2$. We say that $(\mathcal{A}_2, f_2, \mathcal{L}_2)$ is an extension of $(\mathcal{A}_1, f_1, \mathcal{L}_1)$ if $\mathcal{A}_2 \supseteq \mathcal{A}_1$, $\mathcal{L}_2$ is a δ-field extension of $\mathcal{L}_1$ and $f_2|_{\mathcal{A}_1} = f_1$. Clearly extension is a partial order on $\mathbf{H}_{\mathcal{K}}$. It is a consequence of Zorn's Lemma that any element in $\mathbf{H}_{\mathcal{K}}$ extends to a maximal one. We call a maximal element of $\mathbf{H}_{\mathcal{K}}$ a **maximal δ-homomorphism of $\mathcal{K}$**.

Definition 5.4.1.

- The domain of a maximal δ-homomorphism of $\mathcal{K}$ is called a **maximal δ-subring of $\mathcal{K}$**.
- A δ-ring is called a **local δ-ring** if it is a local ring and its maximal ideal is a δ-ideal.

Proposition 5.4.2. *Let $(\mathcal{R}, f, \mathcal{L})$ be a maximal element of $\mathbf{H}_{\mathcal{K}}$ then*

1. *$\mathcal{R}$ is a local δ-ring and $\mathfrak{m} := \mathrm{Ker} f$ is the maximal ideal of $\mathcal{R}$.*
2. *$x \in \mathcal{K} \setminus \mathcal{R}$ if and only if $\mathfrak{m}\{x\} = R\{x\}$.*

Proof. (1) As the kernel of a δ-homomorphism, $\mathfrak{m}$ is a δ-ideal. Suppose $x \notin \mathrm{Ker} f$ then we can extend f to the localization of $\mathcal{R}$ at x by sending x^{-1} to $f(x)^{-1}$. By maximality of $\mathcal{R}$, $x^{-1} \in \mathcal{R}$ and so x is a unit. This shows that $(\mathcal{R}, \mathrm{Ker} f)$ is a δ-local ring.

(2) If $\mathfrak{m}\{x\} = \mathcal{R}\{x\}$, then

$$1 = m_1 P_1(x) + \cdots + m_k P_k(x) \tag{5.1}$$

for some $m_i \in \mathfrak{m}$ and $P_i \in \mathcal{R}\{y\}$, $i = 1, \ldots, k$. So if $x \in \mathcal{R}$ then we get the contradiction "$1 = 0$" by applying f to both sides of Equation 5.1.

Conversely, if $\mathfrak{m}\{x\} \neq \mathcal{R}\{x\}$, then there exists a maximal δ-ideal $\mathfrak{m}'$ of $\mathcal{R}\{x\}$ containing $\mathfrak{m}\{x\}$. Regard $k := \mathcal{R}/\mathfrak{m}$ as a subfield of $k' := \mathcal{R}\{x\}/\mathfrak{m}'$ and we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{R}\{x\} & \longrightarrow & k' & \longrightarrow & \mathcal{L}' \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{R} & \longrightarrow & k & \longrightarrow & \mathcal{L} \end{array}$$

where $\mathcal{L}'$ is a sufficiently saturated δ-closed field such that k' and $\mathcal{L}$ can be embedded into $\mathcal{L}'$ over k. By maximality of $\mathcal{R}$, we have $\mathcal{R}\{x\} = \mathcal{R}$ hence $x \in \mathcal{R}$. $\square$

Now we set $\mathfrak{L} := \mathfrak{L}_\delta(\mathcal{F})$, $T := Th_{\mathfrak{L}}(\mathcal{F})$. A model, $\mathcal{K}$, of T is simply a differentially closed field extending $\mathcal{F}$. An $\mathfrak{L}$-substructure of $\mathcal{K}$ is a δ-subring of $\mathcal{K}$ containing $\mathcal{F}$ and an $\mathfrak{L}$ homomorphism f is simply a δ-homomorphism with domain of f containing $\mathcal{F}$.

Theorem 5.4.3 (Valuative criterion for differential completeness).

Let X be a δ-closed subset of $\mathbb{A}^n$. Then the following are equivalent:

1. *X is δ-complete.*

2. For any $\mathcal{K} \models T$ and any, $\mathcal{R}$, maximal δ-subring of $\mathcal{K}$ containing $\mathcal{F}$, we have $X(\mathcal{K}) = X(\mathcal{R})$.

Proof. $(2 \Rightarrow 1)$ Let Z be an arbitrary δ-closed subset of $X \times \mathbb{A}^m$. We have to show that $\pi_2(Z)$ is δ-closed. Suppose we are given $\mathcal{K}$, $f : \mathcal{A} \to \mathcal{L}$ as in Proposition 5.3.1 and a tuple $\bar{a}$ in $\mathcal{A}$ such that $\mathcal{K} \models \bar{a} \in \pi_2 Z$. Then there exists $\bar{x} \in X(\mathcal{K})$ such that $\mathcal{K} \models (\bar{x}, \bar{a}) \in Z \wedge \bar{x} \in X$. Extend f to a maximal δ-homomorphism $\tilde{f} : \mathcal{R} \to \mathcal{L}'$. Without loss of generality, we can assume $\mathcal{L}'$ is also differentially closed. Note that $\mathcal{R} \supseteq \mathrm{dom}(f) \supseteq \mathcal{F}$, so by (2) $\bar{x}$ is in fact a tuple in $\mathcal{R}$. Since both X and Z are affine δ-closed sets (defined by positive quantifier free formulas), $\mathcal{L}' \models (\tilde{f}(\bar{x}), \tilde{f}(\bar{a})) \in Z \wedge \tilde{f}(x) \in X$. That means $\mathcal{L}' \models \tilde{f}(\bar{a}) \in \pi_2 Z$. Since $\tilde{f}(\bar{a}) = f(\bar{a})$, by model completeness $\mathcal{L} \models f(\bar{a}) \in \pi_2 Z$. And we conclude that $\pi_2 Z$ is δ-closed by Proposition 5.3.1.

$(1 \Rightarrow 2)$ Suppose (2) does not hold. That means there exists $f : \mathcal{R} \to \mathcal{L}$ a maximal δ-homomorphism of $\mathcal{K}$ with $\mathcal{R} \supseteq \mathcal{F}$ and $\bar{x} \in X(\mathcal{K})$ such that some coordinate x_i of $\bar{x}$ is not in $\mathcal{R}$. By Proposition 5.4.2 (1),

$$m_1 P_1(x_i) + \ldots + m_k P_k(x_i) = 1$$

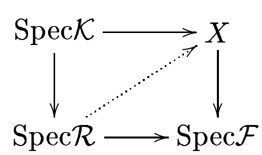
for some $m_1, \ldots, m_k \in \mathfrak{m}$ and $P_1(y), \ldots, P_k(y) \in \mathcal{R}\{y\}$. Now consider the δ-closed subset Z of $X \times \mathbb{A}^k$ defined by the formula $\varphi(\bar{y}, \bar{z})$:

$$\left(\sum_{j=0}^{k} z_j P_j(y_i)\right) - 1 = 0 \wedge \bar{y} \in X.$$

Let $\mathcal{L}$ be the differential closure of $\mathcal{R}/\mathfrak{m}$ and f be the canonical map. Since the restriction of f to $\mathcal{F}$ is a differential field embedding, $\mathcal{L}$ is still a model of T. On one hand, we have $\mathcal{K} \models \exists \bar{y}\ \varphi(\bar{y}, \bar{m})$, since $\bar{x}$ is a witness. On the other hand, the $\mathfrak{m}_i$'s are in the kernel of f hence $\varphi(\bar{y}, f(\bar{m}))$ is the formula "$-1 = 0 \wedge \bar{y} \in X$". So $\mathcal{L} \not\models \exists \bar{y}\ \varphi(\bar{y}, f(\bar{m}))$. By (5.3.1) again, we conclude that $\pi_2 Z$ is not δ-closed and hence X cannot be δ-complete. $\square$

Remark 5.4.4. One should compare Theorem 5.4.3 to the Valuative Criterion of Properness in algebraic geometry (e.g. see [9] Theorem 4.7). In fact, since X is affine, we can rephrase Theorem 5.4.3 in the following way:

For any $\mathcal{K} \models T$, $\mathcal{R}$ maximal δ-subring of $\mathcal{K}$ containing $\mathcal{F}$ and for any commutative square as shown, there exists a unique map from Spec$\mathcal{R}$ to X which makes the diagram commutes.



where the maps are induced by δ-homomorphisms between the δ-rings.

5.5 Kolchin's Examples: $\mathbb{P}^n(\mathcal{C})$ is Differentially Complete

All the results in this section are due to Kolchin. However, the proofs we give here are slightly different, mainly because we will make use of the model theoretic lemma (5.3.1) and results in Section 5.4.

We are going to prove that $\mathbb{P}^n(\mathcal{C})$ is δ-complete. But first we need the following result from commutative algebra ([2] Lemma 5.20).

Lemma 5.5.1. *Let x be a non-zero element of a field K. Let $(R, \mathfrak{m})$ be a local ring in K. Let $R[x]$ be the subring of K generated by x over R and let $\mathfrak{m}[x]$ be the extension of $\mathfrak{m}$ in $R[x]$. Then either $\mathfrak{m}[x] \neq R[x]$ or $\mathfrak{m}[x^{-1}] \neq R[x^{-1}]$.*

Lemma 5.5.1 guarantees that we can always extend our δ-homomorphism to a constant point.

Proposition 5.5.2. *Let $\mathcal{K}, \mathcal{L}$ be δ-fields. Let x be a constant point in $\mathcal{K}$. Let $f : \mathcal{A} \to \mathcal{L}$ be a δ-homomorphism from a δ-subring $\mathcal{A}$ of $\mathcal{K}$ into $\mathcal{L}$. If $\mathcal{C}_{\mathcal{L}}$, the field of constants of $\mathcal{L}$, is algebraically closed then f extends to a map from either $\mathcal{A}\{x\}$ or $\mathcal{A}\{\frac{1}{x}\}$ to $\mathcal{L}$.*

Proof. The case when $x = 0$ is trivial. So assume $x \neq 0$. Let $\tilde{f} : \mathcal{R} \to \mathcal{L}'$ be a maximal extension of f. Since x is a constant $\mathfrak{m}\{x\} = \mathfrak{m}[x]$ and $\mathfrak{m}\{x^{-1}\} = \mathfrak{m}[x^{-1}]$. By Lemma 5.5.1, at least one of these is not the unit ideal. Hence, by Proposition 5.4.2, (2) either x or x^{-1} is in $\mathcal{R}$. Without loss of generality, suppose $x \in \mathcal{R}$. This already shows that f can always be extended either to $\mathcal{A}\{x\}$ or $\mathcal{A}\{x^{-1}\}$ [4]. However, we still have to show that there is an extension of f with range inside $\mathcal{L}$. Let $\mathcal{B} = f(\mathcal{A})$ and $c = \tilde{f}(x)$. Consider the maps $\mathcal{A}[x] \to \mathcal{B}[c]$ obtained by restricting $\tilde{f}$ to $\mathcal{A}[x] (= \mathcal{A}\{x\})$ and $\mathcal{B}[c] \to \mathcal{L}$ by sending c to 0. If c is transcendental over $\mathcal{B}$ then their composition extends f. Otherwise c is algebraic over $\mathcal{B}$ and hence over $\mathcal{L}$. Suppose $y^n + l_1 y^{n-1} + \cdots + l_n$ is the minimal polynomial of c over $\mathcal{L}$. Apply δ to $c^n + l_1 c^{n-1} + \cdots + l_n = 0$ and we get $\delta(l_1)c^{n-1} + \cdots + \delta(l_n) = 0$. Hence by minimality of n, $\delta(l_1) = \cdots = \delta(l_n) = 0$.

[4] Using model completeness, this weaker version is enough to prove Theorem 5.5.4.

Therefore c is actually algebraic over $\mathcal{C}_{\mathcal{L}}$ which is algebraically closed by the assumption. Thus $c \in \mathcal{C}_{\mathcal{L}}$ and this shows that the range of the extension can always be taken inside $\mathcal{L}$. □

We keep the same notations as in Proposition 5.5.2.

Corollary 5.5.3. *Let $f : \mathcal{A} \to \mathcal{L}$ be a δ-homomorphism and let $c_0, \dots, c_n \in \mathcal{K}$ not all are zero such that $\frac{c_i}{c_j} \in \mathcal{C}$ whenever $c_j \neq 0$. Then there exists $0 \leq j \leq n$ with $c_j \neq 0$ such that f can be extended to a δ-homomorphism $\mathcal{A}\{\frac{c_0}{c_j}, \dots, \frac{c_n}{c_j}\} \to \mathcal{L}$.*

Proof. We have the following two cases:

Case 1: Exactly one of the $c_i \neq 0$.

In this case $\mathcal{A}\{\frac{c_0}{c_i}, \dots, \frac{c_n}{c_i}\} = \mathcal{A}$ so the result is trivial.

Case 2: At least two of the c_i are nonzero.

Without loss of generality $c_n \neq 0$ and $c_i \neq 0$ for some $0 \leq i \leq n-1$. By induction on n, we can assume f extends to $g : \mathcal{A}\{\frac{c_0}{c_i}, \dots, \frac{c_{n-1}}{c_i}\} = \mathcal{B} \to \mathcal{L}$. By Corollary 5.5.3, g extends to $\mathcal{B}\{\frac{c_n}{c_i}\} = \mathcal{A}\{\frac{c_0}{c_i}, \dots, 1, \dots, \frac{c_n}{c_i}\}$ or $\mathcal{B}\{\frac{c_i}{c_n}\} = \mathcal{A}\{\frac{c_0}{c_n}, \dots, \frac{c_{n-1}}{c_n}, \frac{c_i}{c_n}\} \supseteq \mathcal{A}\{\frac{c_0}{c_n}, \dots, \frac{c_{n-1}}{c_n}, 1\}$. This shows that we can extends f to the required domain in any case. □

Theorem 5.5.4. *For any $n \in \mathbb{N}$, $\mathbb{P}^n(\mathcal{C})$ is δ-complete.*

Proof. Suppose $Z \subset \mathbb{P}^n(\mathcal{C}) \times \mathbb{A}^m$ is a δ-closed set defined by

$$P_1(y_0, \dots, y_n; z_1, \dots, z_m) = \cdots = P_k(y_0, \dots, y_n; z_1, \dots, z_m) = 0$$

where $P_i(\bar{y}, \bar{z}) \in \mathcal{F}\{\bar{y}, \bar{z}\}$ is δ-homogeneous in $\bar{y}$ for each $1 \leq i \leq k$. Then $\pi_2 Z$ is defined by $\exists \bar{y}\ \varphi(\bar{y}, \bar{z})$, where $\varphi(\bar{y}, \bar{z})$ is the formula

$$\bigwedge_{i=1}^{k} P_i(\bar{y}, \bar{z}) = 0 \wedge \bigvee_{j=0}^{n} \left(y_j \neq 0 \wedge \bigwedge_{i=1}^{n} \delta(y_i) y_j - y_i \delta(y_j) = 0 \right).$$

Again suppose we are given $\mathcal{K}, \mathcal{L}, f : \mathcal{A} \to \mathcal{L}$ and $\bar{a}$ as in (5.3.1). Then there exists $\bar{c} \in \mathcal{K}^{n+1}$ such that not all the c_i's are zero and $\frac{c_i}{c_j}$ is a constant whenever $c_j \neq 0$ and $P_i(c_0, \dots, c_n, \bar{a}) = 0$ for $1 \leq i \leq k$. By Corollary 5.5.3, f extends to $\mathcal{A}\{\frac{c_0}{c_j}, \dots, \frac{c_n}{c_j}\}$ for some $0 \leq j \leq n$. By δ-homogeneity, $P_i(\frac{c_0}{c_j}, \dots, 1, \dots \frac{c_n}{c_j}, \bar{a}) = 0$. Now it is easy to see that $(f(\frac{c_0}{c_j}), \dots, 1, \dots, f(\frac{c_n}{c_j}))$ is a solution to $\varphi(\bar{y}, f(\bar{a}))$ in $\mathcal{L}$. □

5.6 New Examples

So far all the examples we saw are related to the field of constants. From the point of view of model theory, it is natural to ask **whether all the δ-complete sets are nonorthogonal to the constants**. We will give a negative answer to this question.

Definition 5.6.1. Let $\mathcal{A}$ be a δ-subring of $\mathcal{K}$. An element of $\mathcal{A}\{y\}$ is **monic** if it is of the form $y^n + f(y)$ where the total degree of f (as an element in $\mathcal{A}\{y\} = \mathcal{A}[y, \delta y, \delta^2 y, \dots])$ is less than n. An element $x \in \mathcal{K}$ is **monic over** $\mathcal{A}$ if x is a zero of some monic δ-polynomial over $\mathcal{A}$.

We will need the following result of P. Blum [3] in the form that appears in [27].

Proposition 5.6.2 (Blum).

Let $(\mathcal{R}, \mathfrak{m})$ be a maximal δ-subring of $\mathcal{K}$. Then $x \in \mathcal{K}$ is monic over $\mathcal{R}$ $\iff$ $x^{-1} \notin \mathfrak{m}$.

The following theorem gives a family of δ-complete sets. Some members of this family are orthogonal to the field of constants.

Theorem 5.6.3. *Let $P(y)$ be an ordinary polynomial over $\mathcal{F}$. Then the δ-closure in $\mathbb{P}^1$ of the subset defined by $\delta y - P(y) = 0$ is δ-complete.*

Proof. Let X be the δ-closed set defined by $\delta y - P(y) = 0$. First consider the case where $\deg P(y) \leq 1$. Let $P(y) = ay + b$ where a may be zero. Let y_0 be any point of X. Then $X - y_0$ is the set of solution of the equation $\delta z = az$. In any case, even if $a = 0$, the solution set is simply the $\mathcal{C}$-span of z_0 where z_0 is a nonzero solution of the equation. Hence the δ-closure of X in $\mathbb{P}^1$ is isomorphic to $\mathbb{P}^1(\mathcal{C})$ which is δ-complete by (5.5.4). So let us assume $\deg P \geq 2$. By homogenizing the equation, it is easy to see that X is in fact δ-closed in $\mathbb{P}^1$. Now let $x \in X(\mathcal{K})$ and $\mathcal{R}$ be a maximal δ-subring of $\mathcal{K}$ containing $\mathcal{F}$. By Theorem 5.4.3, the assertion is true if we can show that $x \in \mathcal{R}$. By (5.4.2 (2)), it suffices to show that $\mathfrak{m}\{x\} \neq \mathcal{R}\{x\}$. Since $\delta x = P(x)$, $\mathfrak{m}\{x\} = \mathfrak{m}[x]$ and $\mathcal{R}\{x\} = \mathcal{R}[x]$. So by Lemma 5.5.1, it is enough to show that $\mathfrak{m}[x^{-1}]$ is the unit ideal. Since $\deg P \geq 2$, $a^{-1}(\delta y - P(y))$ is monic where a is the leading coefficient of $P(y)$. It follows from Proposition 5.6.2 that $x^{-1} \notin \mathfrak{m}$. Therefore, if x^{-1} is in $\mathcal{R}$, so is x. So let us assume that $x^{-1} \notin \mathcal{R}$. By (5.4.2 (2)) again, we conclude that $\mathfrak{m}\{x^{-1}\}$ is the unit ideal. Moreover, we have $\delta(x^{-1}) = -x^2 P(x)$. This implies that $\mathfrak{m}\{x^{-1}\}$ is a subset of $\mathfrak{m}[x^{-1}, x]$. So there is a nontrivial expression of 1 as:

$$1 = w_{-r}\frac{1}{x^r} + \cdots + w_{-1}\frac{1}{x} + w_0 + w_1 x + \cdots + w_s x^s \qquad (5.2)$$

where $w_i \in \mathfrak{m}$. Note that $1 - w_0$ is a unit in $\mathcal{R}$, so we can move w_0 to the other side of the equation and then divides both sides by $1 - w_0$ and we get the equation

$$1 = m_{-r}\frac{1}{x^r} + \cdots + m_{-1}\frac{1}{x} + m_1 x + \cdots + m_s x^s \quad (m_i \in \mathfrak{m}) \tag{5.3}$$

Applying δ to both sides of Equation 5.3 we get

$$\begin{aligned} 0 = {} & \delta(m_{-r})\frac{1}{x^r} + r m_r \frac{1}{x^{r-1}}\Big(-\frac{P(x)}{x^2}\Big) + \cdots + \delta(m_{-1})\frac{1}{x} + \\ & + m_{-1}\Big(-\frac{P(x)}{x^2}\Big) + \cdots + \delta(m_s)x^s + s m_s x^{s-1}P(x) \\ 0 = {} & \text{ lower degree terms } + s a m_s x^{s+d-1} \end{aligned} \tag{5.4}$$

where $d = \deg P \geq 2$. The important point here is that the coefficient of the highest degree term is a unit times m_s. Divides Equation 5.4 by sax^{d-1} and eliminates $m_s x^s$ with Equation 5.3 we get

$$1 = n_{-t}\frac{1}{x^t} + \cdots + n_0 + \cdots + n_k x^k \tag{5.5}$$

where $k < s$. Since $\mathfrak{m}$ is a δ-ideal, (cf. 5.4.2 (1)) all the n_i are still in $\mathfrak{m}$. Again by performing the same trick to Equation 5.5 as we did to Equation 5.2, we can assume $n_0 = 0$. Thus we get a similar expression for 1 with degree less than s. So by iterating this process, we conclude that $\mathfrak{m}[x^{-1}]$ is the unit ideal. This finishes the proof. $\square$

Remark 5.6.4. If the degree of P is equal to 2, then $\delta y = P(y)$ is a Riccati equation. In [5], Phyllis Cassidy points out that for each Riccati equation there is a projective transformation of $\mathbb{P}^1$ which maps the corresponding variety onto $\mathbb{P}^1(\mathcal{C})$. So in particular we see that the Riccati varieties are δ-complete by (5.5.4) also.

Example 5.6.5. Let X be the δ-closed set defined by $\delta y = y^3 - y^2$. It is δ-complete, strongly minimal and orthogonal to the constants. We will sketch the proof here and refer our readers to [24] for the details. By Theorem 5.6.3, we know that X is δ-complete. Since X is defined by an order one equation, $\mathrm{RM}(X) = 1$ by Corollary 5.5 in [24]. Note also that the δ-ideal $\mathcal{I}$ generated by $\delta y = y^3 - y^2$ is prime since the quotient of $\mathcal{F}\{y\}$ by $\mathcal{I}$ is clearly isomorphic to $\mathcal{F}[y]$. From this we conclude that X is δ-irreducible. So X is strongly minimal.

Let $\mathcal{K}$ be the differential closure of $\mathbb{Q}$. In particular, $\mathcal{C}_{\mathcal{K}}$, the field of constants of $\mathcal{K}$ is algebraically closed. Let a be a generic point of X over $\mathcal{K}$. Then a is transcendental over $\mathcal{K}$. Let $\mathcal{L}$ be the differential closure of $\mathcal{K}\langle a \rangle$. By Lemma 7.3 in [24], to show that X is orthogonal to the constants all we have to do is to show that $\mathcal{L}$ has no new constants; that is $\mathcal{C}_{\mathcal{L}} = \mathcal{C}_{\mathcal{K}}$. By (6.2) and (6.12) in [24], we have $\mathcal{C}_{\mathcal{K}(a)} = \mathcal{C}_{\mathcal{K}}$. By (2.11) in [24], $\mathcal{C}_{\mathcal{L}} = \mathcal{C}_{\mathcal{K}\langle a \rangle}$. Finally since a satisfies $\delta y = y^3 - y^2$, so $\mathcal{K}\langle a \rangle = \mathcal{K}(a)$. Putting all these together, we have $\mathcal{C}_{\mathcal{L}} = \mathcal{C}_{\mathcal{K}}$ and this completes the proof.

APPENDICES

Appendix A

AN EXAMPLE

Recently Hrushovski and Scanlon have shown that in DCF_0 Morley rank and Lascar rank are different (see [11]). By Corollary 2.2.5, we know that $\mathrm{RM} \neq \mathrm{RC}$ in DCF_0 as well. I would like to thank Ehud Hrushovski for pointing out to me the following example which shows that Morley rank and Lascar rank of a type can be different even if the Lascar rank of the type is a limit ordinal. I also benefited from discussing this example with David Marker and Tom Scanlon.

Example A.0.6. Let a be δ-transcendental over $\mathbb{Q}$. Let E_a be the elliptic curve with j-invariant a. Pick b such that $q = \mathrm{tp}(b/a)$ is the generic type of the Manin kernel of E_a.

Claim 1 $\mathrm{RU}(a,b) = \omega$.

It suffices to show that if $\{a,b\}$ forks over some $\mathcal{L}$, then $\mathrm{RU}(a,b/\mathcal{L})$ is finite. Suppose $a \perp \mathcal{L}$, then $tp(\mathcal{L}/a)$ is a nonforking extension of $tp(\mathcal{L}/\emptyset)$. By Proposition 2.8 in [12], q is orthogonal to $tp(\mathcal{L}/\emptyset)$ and hence to $tp(\mathcal{L}/a)$. Since b realizes q, this means $b \perp_a \mathcal{L}$. Putting this together with $a \perp \mathcal{L}$, we have $ab \perp \mathcal{L}$ contradicting our assumption. Therefore $a \not\perp \mathcal{L}$ and hence $\mathrm{RU}(a/\mathcal{L})$ is finite. Moreover, by Proposition 2.6 in [12] q is strongly minimal hence $\mathrm{RU}(b/\mathcal{L}\langle a\rangle) \leq 1$. Now it follows from the Lascar inequalities that $\mathrm{RU}(a,b/\mathcal{L})$ is finite.

Claim 2 $\mathrm{RM}(a,b) = \omega + 1$.

Let $p(x, y)$ be a nonforking extension of $tp(a, b)$ over some ω-saturated differentially closed field $\mathcal{K}$. By strongly minimality of the Manin kernel, p is isolated by the following collection of formulas:

- x is δ-transcendental over $\mathcal{K}$.
- $\mu(y) = 0$ where μ is the Manin map of E_x.
- y is not algebraic over $\mathcal{K}\langle x \rangle$.

Let Φ be any finite sub-collection of the above formulas containing "$\mu(y) = 0$". Suppose (c, d) is a realization of Φ. Since $\mu(d) = 0$, d is δ-algebraic over c. So if $\mathrm{RM}(c, d) > \omega$, c has to be δ-transcendental over $\mathcal{K}$. By rank considerations, d cannot be algebraic over c. So actually we have $p = tp(c, d/\mathcal{K})$. This shows that $\mathrm{RM}(p) \leq \omega + 1$. On the other hand there are only finitely many inequations in Φ, so Φ can be satisfied by (c, d) where c is δ-transcendental over $\mathcal{K}$ and d is a torsion point of E_c. Such a tuple will have Morley rank ω. Thus p is a limit point of Morley rank ω types. Therefore $\mathrm{RM}(a, b) = \mathrm{RM}(p) = \omega + 1$.

Appendix B

MANIN MAP

The Manin kernels of Abelian varieties are candidate for δ-complete sets. Although we are unable to decide this question here, we found out that the formula of the Manin Map which appears in Manin's paper [22] is 'slightly' incorrect. So in this appendix, we set ourselves to find the explicit formula of the Manin map for the family of elliptic curves

$$E : y^2 = x(x-1)(x-t) \tag{B.1}$$

over the affine t-line, $\mathbb{A}^1_t$. Our discussion will mainly be computational. For a more theoretical treatment, we refer the readers to [16], [6] and [19]. The formula of the Manin map for the family $y^2 = 4x^3 - g_2(t)x - g_3(t)$ appears in [34].

B.1 The Gauss-Manin connection

Let $\frac{d}{dt}$ be the standard derivation on $K := \mathbb{C}(t)$. We extend $\frac{d}{dt}$ to a derivation on the function field $L := \mathbb{C}(E)$, by specifying its action on a transcendence basis of L/K. To carry out the computation, we choose $\{x\}$ to be the transcendence basis and let ∂_x be the unique derivation extending $\frac{d}{dt}$ which sends x to 0. By Equation B.1 we have

$$\partial_x(y) := -\frac{y}{2(x-t)}.$$

We can further extend ∂_x to a derivation on, $\Omega_{L/K}$, the L-module of differential forms of L over K by setting $\partial_x(f dx) = \partial_x(f)dx$.

By going to an open set, S, of the base, we can assume $E|_S \to S$ is a smooth map between smooth varieties. Then following Manin's arguments ([22] Ch.1 §1), one can show that:

1. The diagram

$$\begin{array}{ccc} L & \xrightarrow{d} & \Omega_{L/K} \\ \downarrow{\scriptstyle \partial_x} & & \downarrow{\scriptstyle \partial_x} \\ L & \xrightarrow{d} & \Omega_{L/K} \end{array}$$

   commutes. In particular, d preserves both closed and exact forms.

2. The derivation ∂_x takes 1-forms of the 2nd kind to 1-forms of the 2nd kind.

Hence ∂_x induces a derivation, still call it ∂_x, on the K-vector space

$$H^1_{DR}(E/S) := \frac{\{\text{2nd kind}\}}{\{\text{exact form}\}}.$$

One can check that $\nabla : H^1_{DR} \to \Omega_{K/\mathbb{C}} \otimes_K H^1_{DR}$ given by

$$\nabla(f dx) = dt \otimes \partial_x(f)dx$$

is a connection (the Gauss-Manin connection in our case) on H^1_{DR}. And ∂_x is simply ∇ followed by $\frac{d}{dt} \otimes 1$.

As described in Manin's paper, the above construction is canonical in the following sense: If $\{z\}$ is another transcendence basis of L/K and ∂_z is the unique lifting of $\frac{d}{dt}$ sending z to 0 then for any $\omega \in \Omega^1$ we have

$$(\partial_x - \partial_z)\omega = \text{exact form.}$$

Hence ∂_z and ∂_x will induce the same connection on H^1_{DR}.

We fix a K-basis, $\{\frac{dx}{y}, x\frac{dx}{y}\}$, of H^1_{DR} and compute the connection matrix with respect to this basis. In H^1_{DR} we have,

$$\partial_x\left(\frac{dx}{y}\right) = \frac{1}{2(x-t)}\frac{dx}{y} = (a+bx)\frac{dx}{y}$$

where $a, b \in K$. The coefficients a and b can be determined by finding an $f \in L$ such that $\frac{1}{2(x-t)}\frac{dx}{y} + df$ can be express as K-linear combination of $\frac{dx}{y}$ and $x\frac{dx}{y}$. Note that $(a+bx)\frac{dx}{y}$ has a pole of order 2 at ∞ and $\frac{1}{2(x-t)}\frac{dx}{y}$ has a pole of order 2 at $x = t$. Since applying d to a function will increase the order of a pole by 1, hence we are looking for a function f with polar divisor of the form $t + \infty$. The function $\frac{y}{(x-t)}$ satisfies this requirement and f will be a multiple of it. Computation shows that

$$d\frac{y}{(x-t)} = \left(\frac{x^2 - 2tx + t}{2(x-t)}\right)\frac{dx}{y}.$$

We want to eliminate the denominator, $2(x-t)$, after adding $\frac{1}{2(x-t)}\frac{dx}{y}$. This force us to pick f to be $\frac{1}{t(t-1)}\frac{y}{(x-t)}$ and

$$\begin{aligned}\partial_x\left(\frac{dx}{y}\right)+df &= \frac{1}{2(x-t)}\frac{dx}{y}+\left(\frac{x^2-2xt+t}{2t(t-1)(x-t)}\right)\frac{dx}{y}\\ &= \left(\frac{x-t}{2t(t-1)}\right)\frac{dx}{y}.\end{aligned}$$

Therefore

$$a=-\frac{1}{2(t-1)},\quad b=\frac{1}{2t(t-1)}.$$

A similar calculation shows that in H^1_{DR}

$$\partial_x\left(x\frac{dx}{y}\right)=-\frac{1}{2(t-1)}\frac{dx}{y}+\frac{1}{2(t-1)}x\frac{dx}{y}.$$

Hence the connection matrix is

$$\tfrac{1}{2(t-1)}\begin{pmatrix}-1 & t^{-1}\\ -1 & 1\end{pmatrix} \tag{B.2}$$

B.2 The Picard Fuchs equation

To simplify the notation, let $\omega = \frac{dx}{y}$ and $\partial = \partial_x$. From the calculations in the previous section, we have equations:

$$\partial(\omega) = -\frac{\omega}{2(t-1)} + \frac{x\omega}{2t(t-1)} - \frac{1}{t(t-1)}d\Big(\frac{y}{x-t}\Big) \tag{B.3}$$

$$\partial(x\omega) = -\frac{\omega}{2(t-1)} + \frac{x\omega}{2(t-1)} - \frac{1}{t-1}d\Big(\frac{y}{x-t}\Big) \tag{B.4}$$

Since $\dim_K H^1_{DR} = 2$, therefore the classes $\omega, \partial(\omega)$ and $\partial^2(\omega)$ have to be K-linearly dependent. We get the relation by eliminating x from the above equations. Multiply Equation B.3 by $2t(t-1)$ then apply ∂ to the resulting equation. Substitute $\partial(xw)$ using Equation B.4 and we get

$$\begin{aligned}(4t-2)\partial(\omega) + 2t(t-1)\partial^2(\omega) = -t\partial(\omega) - \omega - \frac{\omega}{2(t-1)} + \frac{x\omega}{2(t-1)} \\ - \frac{1}{t-1}d\frac{y}{x-t} - 2\partial\Big(d\frac{y}{x-t}\Big).\end{aligned} \tag{B.5}$$

Again using Equation B.4 we eliminate $\frac{x\omega}{2(t-1)}$ in Equation B.5. Also since d and ∂ commutes we have $2\partial(d\frac{y}{x-t}) = d\frac{y}{(x-t)^2}$. The L.H.S. of Equation B.5 equals

$$\begin{aligned} & -t\partial(\omega) - \omega - \frac{\omega}{2(t-1)} + t\partial(\omega) + \frac{t\omega}{2(t-1)} + \\ & \qquad \frac{1}{t-1}d\frac{y}{x-t} - \frac{1}{t-1}d\frac{y}{x-t} - d\frac{y}{(x-t)^2} \\ & = -\frac{\omega}{2} - d\frac{y}{(x-t)^2}. \end{aligned} \tag{B.6}$$

From this we obtain the well-known Picard-Fuchs equation of our family

$$4t(t-1)\partial^2(\omega) + 4(2t-1)\partial(\omega) + \omega = -2d\frac{y}{(x-t)^2}. \tag{B.7}$$

B.3 The Manin Map

Let F be an algebraic extension of (K, ∂_t). We are going to define the Manin map $M : E(F) \to F$ which will also be a group homomorphism between $E(F)$ and F.

Pick a point $p(t) = (x(t), y(t)) \in E(F)$. By regarding t as a complex variable we can think of $p(t)$ as a section of our family. Let 0 be the zero section. It picks, for each $z \in \mathbb{C}$, the point at infinity of the curve E_z which is the zero element of the group law on E_z. The integral

$$\int_0^{p(t)} \frac{dx}{y}$$

is well define up to the lattice generated by

$$\pi_1 := \int_{\gamma_1} \frac{dx}{y}, \quad \pi_2 := \int_{\gamma_2} \frac{dx}{y}$$

where γ_1 and γ_2 are cycles representing the generators of the homology group $H_1(E_z, \mathbb{C})$.

Let Λ be the differential operator

$$2t(t-1)\partial^2 + 2(2t-1)\partial + \frac{1}{2}$$

and z varies through the t-line. From the Picard-Fuchs equation (Equation B.7) we get $\Lambda\omega = -\frac{y}{(x-t)^2}$. Since Λ commutes with integration over closed cycles, $\Lambda(\pi_i(t)) = 0$. Hence the Manin map

$$p(t) \overset{M}{\longmapsto} \Lambda \int_0^{p(t)} \frac{dx}{y}$$

is well defined. The fact that it is a group homomorphism follows from linearity of Λ and Abel's theorem.

Now we can set forth the computation of M. Regard x and t as two independent complex variables. Let $y(x,t)$ be a branch of $\sqrt{x(x-1)(x-t)}$. For simplicity of notation, we write x' for $\partial(x(t))$, y for $y(t) = y(x(t), t)$, etc. By Fundamental Theorem of Calculus,

$$\partial \int_0^{x(t)} \frac{dx}{y} = \frac{x'}{y} + \int_0^{x(t)} \partial\Big(\frac{1}{y}\Big) dx.$$

Hence

$$\partial^2 \int_0^{x(t)} \frac{dx}{y} = \partial\Big(\frac{x'}{y} + \int_0^{x(t)} \partial\Big(\frac{1}{y}\Big) dx\Big)$$
$$= \partial\Big(\frac{x'}{y}\Big) + \frac{x'}{2(x-t)y} + \int_0^{x(t)} \partial^2\Big(\frac{1}{y}\Big) dx. \tag{B.8}$$

Thus

$$M(p(t)) = \Lambda \int_0^{x(t)} \frac{dx}{y}$$
$$= 2t(t-1)\partial\Big(\frac{x'}{y}\Big) + \frac{t(t-1)x'}{(x-t)y} + 2(2t-1)\frac{x'}{y} - \int_0^{x(t)} \Lambda\frac{dx}{y}.$$

Note that $\frac{y}{(x-t)^2}$ has a simple zero at infinity, therefore

$$M(p(t)) = -\frac{y}{(x-t)^2} + \Big(2t(t-1)\frac{x'}{y}\Big)' + \frac{t(t-1)x'}{(x-t)y} \tag{B.9}$$

and this is the explicit formula of the Manin Map we set out to find.

B.4 An application

Let K be an algebraically closed field of characteristic 0 and t be a transcendental element over K. We make $K(t)$ into a δ-field by setting $\delta a = 0$ for all $a \in K$ and $\delta(t) = 1$. Let $F \supseteq K(t)$ be the algebraic closure of $K(t)$. The elements of F are Puiseux series in t over K, let v be the valuation $v(t^r) = r$ on F. Let E be the curve $y^2 = x(x-1)(x-t)$ defined over $K(t)$. The Manin Map in this case is:

$$M(x,y) = -\frac{y}{(x-t)^2} + \Big(2t(t-1)\frac{x'}{y}\Big)' + \frac{t(t-1)}{(x-t)}\frac{x'}{y}. \tag{B.10}$$

Regarding M as a map from $E(F)$ to F, we verify below that the values of the elements in the image are bounded below by -1. In particular, M is not surjective. It follows from the proof of Theorem 2.1 in [23] that the kernel of M can be realized as a differential Galois group over F.

To show that M is not surjective, we have the following cases to consider:

case I: $v(x) < 0$.

Then $v(x-1) = v(x-t) = v(x)$ hence $v(y) = \frac{3}{2}v(x)$ and $v(\frac{x'}{y}) = v(x') - v(y) = v(x) - 1 - \frac{3}{2}v(x) = -\frac{v(x)}{2} - 1$. So

$$\begin{aligned} v\Big(\frac{y}{(x-t)^2}\Big) &= v(y) - 2v(x-t) \\ &= \frac{3}{2}v(x) - 2v(x) = -\frac{v(x)}{2} \\ v\Big(2t(t-1)\frac{x'}{y}\Big) &= 1 - \frac{v(x)}{2} - 1 = -\frac{v(x)}{2} > 0 \end{aligned}$$

so

$$v\Big(2t(t-1)\frac{x'}{y}\Big)' = -\frac{v(x)}{2} - 1 > -1$$

and

$$v\Big(\frac{t(t-1)}{(x-t)}\frac{x'}{y}\Big) = 1 - v(x) - \frac{v(x)}{2} - 1 = -\frac{3}{2}v(x)$$

therefore, $v(M) > -1$.

case II: $v(x) > 1$.

Then $v(x-1) = 0, v(x-t) = 1, v(x') = v(x) - 1$ and $v(y) = \frac{1+v(x)}{2}, v(\frac{x'}{y}) = \frac{v(x)-3}{2}$ so

$$v\Big(\frac{y}{(x-t)^2}\Big) = \frac{1+v(x)}{2} - 2 = \frac{v(x)-3}{2}$$

$$v\Big(2t(t-1)\frac{x'}{y}\Big) = 1 + \frac{v(x)-3}{2} > 0$$

so

$$v\Big(2t(t-1)\frac{x'}{y}\Big)' = \frac{v(x)-3}{2}$$

and

$$v\Big(\frac{t(t-1)}{(x-t)}\frac{x'}{y}\Big) = \frac{v(x)-3}{2}$$

therefore, $v(M) \geq \frac{v(x)-3}{2} > -1$.

case III: $0 < v(x) < 1$.

We have $v(x-1) = 0, v(x) = v(x-t)$ so $v(y) = v(x)$ and $v(\frac{x'}{y}) = v(x) - 1 - v(x) = -1$. Also

$$v\Big(\frac{y}{(x-t)^2}\Big) = v(x) - 2v(x) = -v(x)$$
$$v\Big(2t(t-1)\frac{x'}{y}\Big) = 1 - 1 = 0$$

so

$$v\Big(2t(t-1)\frac{x'}{y}\Big)' > -1$$

and

$$v\Big(\frac{t(t-1)}{(x-t)}\frac{x'}{y}\Big) = 1 - v(x) - 1 = -v(x)$$

therefore, $v(M) > -1$.

case IV(a): $v(x) = 0$ **and** $v(x-1) = 0$.

Then $v(x-t)=0, v(x')>-1$ and $v(y)=0, v(\frac{x'}{y})>-1$

$$v\Big(\frac{y}{(x-t)^2}\Big)=0$$
$$v\Big(2t(t-1)\frac{x'}{y}\Big)=1+v(\frac{x'}{y})>0$$

so

$$v\Big(2t(t-1)\frac{x'}{y}\Big)'>-1$$

and

$$v\Big(\frac{t(t-1)}{(x-t)}\frac{x'}{y}\Big)=1+v(\frac{x'}{y})>0$$

therefore, $v(M)>-1$.

case IV(b): $v(x)=0$ and $v(x-1)>0$.

Then $v(x-t)=0, v(y)=\frac{v(x-1)}{2}$ and $v(x')=v(x-1)-1>-1$ so $v(\frac{x'}{y})>v(x-1)-1-\frac{v(x-1)}{2}=\frac{v(x-1)}{2}-1>-1$ and

$$v\Big(\frac{y}{(x-t)^2}\Big)=v(y)>0$$
$$v\Big(2t(t-1)\frac{x'}{y}\Big)=1+v(\frac{x'}{y})>0$$

so

$$v\Big(2t(t-1)\frac{x'}{y}\Big)' > -1$$

and

$$v\Big(\frac{t(t-1)}{(x-t)}\frac{x'}{y}\Big) = 1 + v(\frac{x'}{y}) > 0$$

therefore, $v(M) > -1$.

case V(a): $v(x) = 1$ **and** $v(x-t) = 1$**.**

Then $v(x-1) = 0, v(y) = 1, v(x') = 0$ and so $v(\frac{x'}{y}) = -1$. Thus

$$v\Big(\frac{y}{(x-t)^2}\Big) = 1 - 2 = -1$$
$$v\Big(2t(t-1)\frac{x'}{y}\Big)' > -1$$
$$v\Big(\frac{t(t-1)}{x-t}\frac{x'}{y}\Big) = -1$$

therefore again, $v(M) \geq -1$.

case V(b): $v(x) = 1$ **and** $v(x-t) > 1$**.**

This is the most tedious case. Here we have $x = t + h$ where $v(h) > 1$. So $y^2 = hx(x-1)$ and $v(y) = \frac{v(h)+1}{2}$. Also we have

$$-\frac{y}{h^2} = -\frac{y^2}{hy} = -\frac{x(x-1)}{hy} = -\Big(\frac{t(t-1)}{hy} + \frac{2t-1}{y} + \frac{h}{y}\Big)$$
$$\frac{t(t-1)}{h}\frac{x'}{y} = \frac{t(t-1)}{hy} + \frac{t(t-1)h'}{hy}$$

therefore

$$-\frac{y}{h^2} + \frac{t(t-1)}{h}\frac{x'}{y} = --\frac{2t-1}{y} - \frac{h}{y} + \frac{t(t-1)h'}{hy}$$

and

$$\Big(2t(t-1)\frac{x'}{y}\Big)' = 2(2t-1)\frac{x'}{y} + 2t(t-1)\Big[-x'\frac{y'}{y^2} + \frac{h''}{y}\Big].$$

From the equation of the elliptic curve, we get

$$2yy' = h(2x-1)x' + h'x(x-1)$$

so

$$\frac{y'}{y} = \frac{(2x-1)x'}{2x(x-1)} + \frac{h'}{2h}.$$

Hence the whole thing is equal to $(yx(x-1))^{-1}$ times

$$-(2t-1)x(x-1) - hx(x-1) - (2x-1)t(t-1)(x')^2 - \frac{(h')^2 t(t-1)}{h} x(x-1)$$
$$+2(2t-1)x'x(x-1) + 2t(t-1)x(x-1)h''.$$

At this point, formally we expand the expression by substituting $x = t + h$. But informally we just set $x = t$ and $x' = 1$; bear in mind that the remaining terms will have valuation greater than that of h. After cancellation we find that whole expression will have value larger than h so the image will have valuation bigger than

$$v\Big(\frac{h}{yx(x-1)}\Big) = v(h) - \frac{v(h)+1}{2} - 1 = \frac{v(h)-3}{2} > -1.$$

Therefore any point in the image of the Manin Map will have value larger than or equal to -1; in particular the map is not surjective!

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- Chains of Differential Subvarieties of an Algebraic Variety. (preprint)

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