Complete sets in differentially closed fields

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Completeness is one of the most fundamental notions in algebraic geometry. This article is an attempt to understand the "complete sets" in the case of differential algebraic geometry. The methods that we use here come from both model theory and differential algebra. The model theoretic part is taken from a paper by van den Dries [15]. Using what he called a "Lyndon-Robinson" type result, van den Dries gave a proof of the main theorem of classical elimination theory. The completeness of projective varieties easily follows from this.

Differential completeness was studied in different settings by Blum in [2] and by Kolchin in [8]. In [8], Kolchin proved that the set of constant points of a projective space is differentially complete while the whole projective space is not. However, those were the only examples and differentially complete sets have not been extensively studied since then.

Throughout this article, $\mathcal{F}$ is a fixed differentially closed field of characteristic 0 with a unique derivation δ and $\mathcal{C}$ is the field of constants of $\mathcal{F}$. All δ-varieties that we consider are defined over $\mathcal{F}$. The affine n-space and the projective n-space over $\mathcal{F}$ are denoted by $\mathbb{A}^n$ and $\mathbb{P}^n$ respectively. All δ-varieties that we consider are defined over $\mathcal{F}$. Also we identify a δ-variety with its set of $\mathcal{F}$-points. The language for our discussion is $\mathfrak{L}_\delta(\mathcal{F}) = \{0, 1, +, -, \cdot, \delta, c_f : f \in \mathcal{F}\}$ where c_f is a constant symbol for the element f of $\mathcal{F}$. In this context, we show that every δ-complete set is affine and is definably isomorphic to a δ-complete subset of the line $\mathbb{A}^1$. This is exactly "opposite" to the phenomenon that happens in algebraic geometry since a complete variety is never affine unless it is a point. Next we obtain a valuative criterion for δ-completeness. Using this criterion, we are able to obtain Kolchin's result on the differential completeness of $\mathbb{P}^n(\mathcal{C})$. Also we find a new family of δ-complete sets. Some members of this family are orthogonal to the field of constants. Hence, from the model theoretic point of view, these δ-complete sets are quite different from the set of constant points in a projective space.

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1 Differential Completeness and Properties of Differentially Complete Sets

We say a subset X of the affine space $\mathbb{A}^n$ is **δ-closed** if it is the zero set of a collection of δ-polynomials in $\mathcal{F}\{y_1, \ldots, y_n\}$. By the differential basis theorem, we can assume the collection is finite.

Definition 1.1. Let f be a non-constant polynomial in $\mathcal{F}\{y_0, \ldots, y_n\}$. We say that f is **δ-homogeneous of degree** d if

$$f(ty_0, \ldots, ty_n) = t^d f(y_0, \ldots, y_n),$$

for some t δ-transcendental over $\mathcal{F}\{y_0, \ldots, y_n\}$.

Just like ordinary homogeneous polynomials, δ-homogeneous polynomials can be obtained by homogenization. Let f be a δ-polynomial in $y_1, \ldots, y_n$. One can easily check that, for d sufficiently large, $y_0^d f(\frac{y_1}{y_0}, \ldots, \frac{y_n}{y_0})$ is a δ-homogeneous polynomial of degree d. For example $f = \delta(y_1)$, then $y_0^2 \delta(\frac{y_1}{y_0}) = y_0\delta(y_1) - y_1\delta(y_0)$ is a δ-homogeneous polynomial of degree 2.

In general, we say a subset X of an $\mathcal{F}$-variety V is δ-closed if the intersection of X with every affine Zariski open subset of V is an affine δ-closed set. However, using δ-homogeneous polynomial, we can give a more direct characterization for δ-closed subsets of $\mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{A}^m$.

- A subset of $\mathbb{P}^n$ is **δ-closed** if it is the zero set of a collection of δ-homogeneous polynomials in $\mathcal{F}\{y_0, \ldots, y_n\}$.

- A subset of $\mathbb{P}^n \times \mathbb{A}^m$ is **δ-closed** if it is the zero set of a collection of δ-polynomials $\{f_i\}$ in $\mathcal{F}\{y_0, \ldots, y_n, z_1, \ldots, z_m\}$ such that f_i is δ-homogeneous in $\bar{y}$, for each i.

Let us give an example.

Example 1.2. Consider Z, the δ-closed subset of $\mathbb{A}^2$ defined by:

$$z(\delta y)^2 + (y^4 - 1) = 0$$
$$2z\delta^2(y) + \delta z \delta y + 4y^3 = 0.$$

Homogenizing the first equation with respect to y, we get

$$z(y_1 \delta y_0 - y_0 \delta y_1)^2 + y_0^4 - y_1^4 = 0.$$

Note that the δ-closed set defined by this equation does not intersect the line $[1:0] \times \mathbb{A}^1$. Hence Z is actually δ-closed in $\mathbb{P}^1 \times \mathbb{A}^1$.

The following notion is the differential counterpart of completeness in algebraic geometry.

Definition 1.3. A δ-closed set $X \subseteq \mathbb{P}^n$ is **δ-complete** if the second projection $\pi_2 : X \times Y \to Y$ is a δ-closed map for every quasiprojective δ-variety Y.

It is easy to see that X is δ-complete if and only if all its δ-irreducible components are, so we may assume X is δ-irreducible.

The next proposition records some properties of δ-complete sets that we are going to use later.

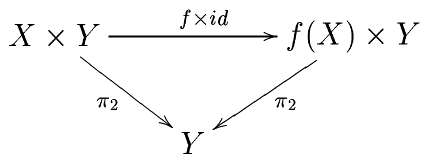
Proposition 1.4. *Let X be a δ-complete set and Y be a quasiprojective δ-variety.*

1. *Let $f : X \to Y$ be continuous in the δ-topology. Then $f(X)$ is δ-closed in Y and is δ-complete.*

2. *Any δ-closed subset of X is δ-complete.*

3. *If Y is another δ-complete set, then so is $X \times Y$.*

Proof. (1) Suppose $Y \subseteq \mathbb{P}^m$. We may view f as a map from X to $\mathbb{P}^m$ which factors through Y. If $f(X)$ is δ-closed in $\mathbb{P}^m$, it is δ-closed in Y. Replacing Y by $\mathbb{P}^m$, we consider the map $f \times id \colon X \times \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^m$. Note that the graph of f is the inverse image of the diagonal of $\mathbb{P}^m \times \mathbb{P}^m$ which is δ-closed. Therefore the graph of f is δ-closed and by the δ-completeness of X so is $f(X) = \pi_2(\text{graph of } f)$. Moreover, $f(X)$ is δ-complete because the following diagram commute:



(2) Let Z be a δ-closed subset of X. The assertion holds because X is δ-complete and the projection $Z \times Y \to Y$ factors through the inclusion $Z \times Y \to X \times Y$ which is a δ-closed map.

(3) The projection $X \times Y \times Z \to Z$ factors through $Y \times Z \to Z$. □

Remark 1.5. Note that whether a subset is δ-closed is a local question, so by a standard reduction argument (cf [14] Chapter 1, §5, Theorem. 3), **"X is δ-complete" is equivalent to:** $\forall m \in \mathbb{N}$, $\pi_2 \colon X \times \mathbb{A}^m \to \mathbb{A}^m$ **is a δ-closed map.**

2 Differentially Complete Sets are Affine

There is a fundamental difference between δ-completeness and completeness in algebraic geometry: **Projective spaces are complete but not δ-complete**. In [8], Kolchin gives a family of examples which shows that $\mathbb{P}^n$ is not δ-complete for any $n \geq 1$. Here we will give Kolchin's example for $\mathbb{P}^1$ explicitly by exhibiting the defining equations of a δ-closed set in $\mathbb{P}^1 \times \mathbb{A}^1$ and argue that the projection is not δ-closed.

Proposition 2.1. $\mathbb{P}^n$ *is not δ-complete.*

Proof. We have shown, in example 1.2, that the set Z defined by

$$z(\delta y)^2 + (y^4 - 1) = 0 \tag{1}$$

$$2z\delta^2 y + \delta z \delta y + 4y^3 = 0. \tag{2}$$

is a δ-closed subset of $\mathbb{P}^1 \times \mathbb{A}^1$. We will argue that $\pi_2 Z$ is not δ-closed in $\mathbb{A}^1$.

First note that equation (2) is obtained by differentiating equation (1) and then dividing by δy. So any solution, (y_0, z_0) of equation (1) with $\delta y_0 \neq 0$ is a solution to the whole system. Let b be δ-transcendental over $\mathcal{F}$. By the axioms of differentially closed field, we can find a, in some differentially closed field containing $\mathcal{F}\langle b \rangle$, a solution to the following system:

$$b(\delta y)^2 + (y^4 - 1) = 0$$
$$y^4 - 1 \neq 0.$$

In particular, $\delta a \neq 0$ and hence (a, b) is a point in Z. Thus $b \in \pi_2 Z$ is dense in the δ-topology of $\mathbb{A}^1$. However, 0 is not in $\pi_2 Z$, since $y^4 - 1$ and $4y^3$ have no common zeros. So $\pi_2 Z$ is not δ-closed. This shows that $\mathbb{P}^1$ is not δ-complete. Finally since $\mathbb{P}^{n-1}$ can be imbedded as a δ-closed subset of $\mathbb{P}^n$, so by Proposition 1.4 (2), we conclude that $\mathbb{P}^n$ is not δ-complete. □

Using the fact that $\mathbb{P}^n$ is not δ-complete, we argue that δ-complete sets are "small"—that means having finite U-rank. For the definition of U-rank, readers can consult Chapter 4 of [9]. Also, [13] contains a summary of properties of U-rank that we use here.

Let $\mathcal{K} \supset \mathcal{F}$ be an ω-saturated differentially closed field.

Proposition 2.2. *Let X be a proper δ-closed subset of $\mathbb{P}^n$ and $\mathbf{p} \in \mathbb{P}^n \setminus X$. Suppose $\mathrm{RU}(X) \geq \omega$. Then $\mathrm{RU}(\pi_{\mathbf{p}}(X)) \geq \omega$.*

Proof. Let $\mathbf{a} \in \mathcal{K}$ be a generic point of X and $\mathbf{b} = \pi_{\mathbf{p}}(\mathbf{a})$. Then we have $\mathbf{a} \in X \cap \overline{\mathbf{pb}}$. The intersection does not contain $\mathbf{p}$ so it is a proper δ-closed subset of the projective line $\overline{\mathbf{pb}}$. Hence $\mathrm{RU}(\mathbf{a}/\mathbf{b})$ is finite. By the Lascar inequalities,

$$\omega \leq \mathrm{RU}(\mathbf{a}) \leq \mathrm{RU}(\mathbf{a}, \mathbf{b}) \leq \mathrm{RU}(\mathbf{a}/\mathbf{b}) \oplus \mathrm{RU}(\mathbf{b})$$

therefore we have $\omega \leq \mathrm{RU}(\mathbf{b}) \leq \mathrm{RU}(\pi_{\mathbf{p}}(X))$. □

Theorem 2.3. *Every δ-complete set is of finite U-rank.*

Proof. Suppose X is an infinite U-rank δ-complete set in $\mathbb{P}^n$. By Proposition 2.2 and 1.4 (1), the projections of X will have infinite U-rank and are δ-complete. Since $\mathbb{P}^1$ is irreducible in the δ-topology and $\mathrm{RU}(\mathbb{P}^1) = \omega$, any infinite U-rank δ-closed subset of $\mathbb{P}^1$ must be the whole space. So by repeatedly projecting X to lower dimensional spaces, we conclude that the image must be the whole projective space which is not δ-complete by Proposition 2.1. This is a contradiction. □

Now we can show that every δ-complete set in projective space is actually sitting inside an affine Zariski open set. In fact, we have

Corollary 2.4. *Every δ-complete subset is definably isomorphic to an affine δ-closed set. In fact, it is definably isomorphic to a δ-complete set in $\mathbb{A}^1$.*

Proof. Let X be δ-complete. By Theorem 2.3, $\mathrm{RU}(X) < \omega$. Then by Proposition 1.1 and Theorem 1.7 in [13], we know that

> $\mathcal{K} \models$ "X is contained in the complement of a hyperplane and there exists a definable isomorphism from X into $\mathbb{A}^1$".

So by model completeness of the theory of differentially closed fields, the same statement is true in $\mathcal{F}$. Finally, since the definable isomorphism is a composition of projections, in particular it is continuous with respect to the δ-topology, therefore by Proposition 1.4 (1) the image is δ-complete as well. $\square$

3 A Model Theoretic Fact

Using the results from the previous section, we can, and will, assume that X is an affine δ-closed set. Our next goal is to derive a general test to decide if a given δ-closed set is δ-complete. Our approach to this problem is quite näive, namely verify the definition directly. For each $m \in \mathbb{N}$, we pick an arbitrary δ-closed subset of $X \times \mathbb{A}^m$ and argue that its projection is δ-closed. The reason why model theoretic methods are useful to us is that the basic geometric objects, in this case affine δ-closed sets, are definable. In fact, an affine δ-closed set over $\mathcal{F}$ is nothing but a set that is definable by a positive quantifier free formula in our language. With this observation, showing that a definable set Y is δ-closed is equivalent to showing that $\mathcal{F} \models \bar{z} \in Y \leftrightarrow \psi(\bar{z})$ for some positive quantifier free formula ψ. Therefore the first thing we need is a way to tell whether a given formula is equivalent to a positive quantifier free formula. Fortunately, in [15] van den Dries told us one such test.

Proposition 3.1 (van den Dries).
Let T be a complete $\mathfrak{L}$-theory and $\phi(v_1, \ldots, v_m)$ an $\mathfrak{L}$-formula [1] *(without parameters), then the following are equivalent:*

[1] In case $m = 0$, we assume that $\mathfrak{L}$ has a constant symbol.

1. *There exists a positive quantifier free formula ψ such that*

$$T \vdash \forall \bar{v}\ \phi(\bar{v}) \leftrightarrow \psi(\bar{v}).$$

2. *For any models K and L of T and each homomorphism $f : A \to L$ from a substructure A of K into L we have:*

$$\text{if } \bar{a} \in A^m \text{ and } K \models \phi(\bar{a}), \text{ then } L \models \phi(f(\bar{a})).$$

4 Valuative Criterion for Differential Completeness

We continue our search for a test for δ-completeness. The result of van den Dries in the previous section ties this problem with the problem of extending δ-homomorphisms. The latter subject was studied by Blum in [3] and Morrison in [11] and [12]. Here we gather a few basic facts and introduce some terminology that we are going to use.

Proposition 4.1. *Let $\mathcal{R}$ be a δ-ring containing $\mathbb{Q}$. Then every δ-ideal in $\mathcal{R}$ is contained in some prime δ-ideal.*

Proof. Given a δ-ideal $\mathcal{I}$ of $\mathcal{R}$. By Zorn's lemma, let $\mathcal{P}$ be a maximal element among the δ-ideals that contain $\mathcal{I}$. Since $\mathcal{R} \supset \mathbb{Q}$, the radical of a δ-ideal in $\mathcal{R}$ is still a δ-ideal (see [7], p.62 or (1.3–1.6) in [4]). So $\mathcal{P}$ is radical. By (1.6) in [4], every radical δ-ideal in $\mathcal{R}$ is an intersection of prime δ-ideals. Therefore by maximalilty of $\mathcal{P}$, it is a prime ideal. □

Let $\mathcal{K}$ be a δ-field, consider the set

$$\begin{aligned}\mathbf{H}_{\mathcal{K}} = \{ & (\mathcal{A}, f, \mathcal{L}) : \mathcal{A} \text{ is a } \delta\text{-subring of } \mathcal{K}, \mathcal{L} \text{ is a } \delta\text{-field and} \\ & f : \mathcal{A} \to \mathcal{L} \text{ is a } \delta\text{-homomorphism.}\}\end{aligned}$$

Let $(\mathcal{A}_i, f_i, \mathcal{L}_i) \in \mathbf{H}_{\mathcal{K}}$, $i = 1, 2$. We say that f_2 extends f_1 if $\mathcal{A}_2 \supseteq \mathcal{A}_1$, $\mathcal{L}_2$ is a δ-field extension of $\mathcal{L}_1$ and $f_2|_{\mathcal{A}_1} = f_1$. We denote this by $(\mathcal{A}_2, f_2, \mathcal{L}_2) \geq (\mathcal{A}_1, f_1, \mathcal{L}_1)$. The relation $\geq$ is a partial order on $\mathbf{H}_{\mathcal{K}}$ and it is a consequence of Zorn's Lemma that any element in $\mathbf{H}_{\mathcal{K}}$ extends to a maximal one. We call a maximal element of $\mathbf{H}_{\mathcal{K}}$ a **maximal δ-homomorphism** of $\mathcal{K}$.

Definition 4.2.

- A δ-subring of $\mathcal{K}$ is called **maximal** if it is the domain of a maximal δ-homomorphism of $\mathcal{K}$.
- A δ-ring is called a **local δ-ring** if it is a local ring and its maximal ideal is a δ-ideal.

Proposition 4.3. *Let* $(\mathcal{R}, f, \mathcal{L})$ *be a maximal element of* $\mathbf{H}_{\mathcal{K}}$*. Then*

1. $\mathcal{R}$ *is a local δ-ring and* $\mathfrak{m} = kerf$ *is the maximal ideal of* $\mathcal{R}$*.*
2. $x \in \mathcal{K} \setminus \mathcal{R} \iff \mathfrak{m}\{x\} = \mathcal{R}\{x\}$

Proof. (1) As the kernel of a δ-homomorphism, $\mathfrak{m}$ is clearly a δ-ideal. Suppose $x \notin \mathrm{ker} f$, then we can extend f to the localization of $\mathcal{R}$ at x by sending x^{-1} to $f(x)^{-1}$. By maximality of $\mathcal{R}$, $x^{-1} \in \mathcal{R}$. Hence x is a unit. This shows that $(\mathcal{R}, \mathfrak{m})$ is a local δ-ring.

(2) If $\mathfrak{m}\{x\} = \mathcal{R}\{x\}$, then 1 can be expressed as

$$1 = m_1 P_1(x) + \cdots + m_k P_k(x)$$

for some $m_i \in \mathfrak{m}$ and $P_i \in \mathcal{R}\{y\}$. So if $x \in \mathcal{R}$, we get the contradiction "1=0" by applying f on both sides of the above equation.

Conversely if $\mathfrak{m}\{x\} \neq \mathcal{R}\{x\}$, then by Proposition 4.1 there is a prime δ-ideal $\mathfrak{m}'$ of $\mathcal{R}\{x\}$ containing $\mathfrak{m}\{x\}$. Let k' be the field of fractions of $\mathcal{R}\{x\}/\mathfrak{m}'$ and $\mathcal{L}'$ be a common differential field extension of k' and $\mathcal{L}$ over k. Then we have the following diagram commute.

$$\begin{array}{ccccc}
\mathcal{R}\{x\} & \longrightarrow & k' & \longrightarrow & \mathcal{L}' \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{R} & \longrightarrow & k & \longrightarrow & \mathcal{L}
\end{array}$$

By maximality of $\mathcal{R}$, we have $x \in \mathcal{R}$. □

Now we set $\mathfrak{L} := \mathfrak{L}_\delta(\mathcal{F})$, $T := Th_{\mathfrak{L}}(\mathcal{F})$. A model, $\mathcal{K}$, of T is simply a differentially closed field extending $\mathcal{F}$. An $\mathfrak{L}$-substructure of $\mathcal{K}$ is a δ-subring of $\mathcal{K}$ containing $\mathcal{F}$ and an $\mathfrak{L}$ homomorphism f is simply a δ-homomorphism fixing $\mathcal{F}$ pointwise.

Theorem 4.4 (Valuative criterion for differential completeness). *Let* X *be a δ-closed subset of* $\mathbb{A}^n$*. Then the following are equivalent:*

1. *X is δ-complete.*

2. *For any $\mathcal{K} \models T$ and any, $\mathcal{R}$, maximal δ-subring of $\mathcal{K}$ containing $\mathcal{F}$, we have $X(\mathcal{K}) = X(\mathcal{R})$.*

Proof. ($2 \Rightarrow 1$) Let Z be an arbitrary δ-closed subset of $X \times \mathbb{A}^m$. We have to show that $\pi_2 Z$ is δ-closed. Suppose we are given $\mathcal{K}$, $f : \mathcal{A} \to \mathcal{L}$ as in Proposition 3.1 and a tuple $\bar{a}$ in $\mathcal{A}$ such that $\mathcal{K} \models \bar{a} \in \pi_2 Z$. Then there exists $\bar{x} \in \mathcal{K}^n$ such that $\mathcal{K} \models (\bar{x}, \bar{a}) \in Z \wedge \bar{x} \in X$. Extend f to a maximal δ-homomorphism $\tilde{f} : \mathcal{R} \to \mathcal{L}'$. By taking differential closure, we can assume $\mathcal{L}'$ is differentially closed. Note that $\mathcal{R} \supseteq \mathcal{A} \supseteq \mathcal{F}$. So by (2), $\bar{x}$ is in fact a tuple in $\mathcal{R}$. Since both X and Z are affine δ-closed sets (defined by positive quantifier free formulas), $\mathcal{L}' \models (\tilde{f}(\bar{x}), \tilde{f}(\bar{a})) \in Z \wedge \tilde{f}(x) \in X$. That means $\mathcal{L}' \models \tilde{f}(\bar{a}) \in \pi_2 Z$. Since $\tilde{f}(\bar{a}) = f(\bar{a})$, by model completeness $\mathcal{L} \models f(\bar{a}) \in \pi_2 Z$. And we conclude that $\pi_2 Z$ is δ-closed by Proposition 3.1.

($1 \Rightarrow 2$) Suppose (2) does not hold. That means there exists $f : \mathcal{R} \to \mathcal{L}$ a maximal δ-homomorphism of $\mathcal{K}$ with $\mathcal{R} \supseteq \mathcal{F}$ and $\bar{x} \in X(\mathcal{K})$ such that some coordinate x_i of $\bar{x}$ is not in $\mathcal{R}$. By Proposition 4.3 (1),

$$m_1 P_1(x_i) + \ldots + m_k P_k(x_i) = 1$$

for some $m_1, \ldots, m_k \in \mathfrak{m}$ and $P_1(y), \ldots, P_k(y) \in \mathcal{R}\{y\}$. Now consider the δ-closed subset Z of $X \times \mathbb{A}^k$ defined by the formula $\varphi(\bar{y}, \bar{z})$:

$$\left(\sum_{j=0}^{k} z_j P_j(y_i)\right) + 1 = 0 \wedge \bar{y} \in X.$$

Let $\mathcal{L}$ be the differential closure of $\mathcal{R}/\mathfrak{m}$ and f be the canonical map. Since the restriction of f to $\mathcal{F}$ is injective, $\mathcal{L}$ is still a model of T. On one hand, we have $\mathcal{K} \models \exists \bar{y}\ \varphi(\bar{y}, \bar{m})$, since $\bar{x}$ is a witness. On the other hand, the m_i's are in the kernel of f hence $\varphi(\bar{y}, f(\bar{m}))$ is the formula "$1 = 0 \wedge \bar{y} \in X$". So $\mathcal{L} \not\models \exists \bar{y}\ \varphi(\bar{y}, f(\bar{m}))$. And now by (3.1) again, we conclude that $\pi_2 Z$ is not δ-closed and hence X is not δ-complete. □

Remark 4.5. One should compare Theorem 4.4 to the valuative criterion of properness in algebraic geometry (c.f. [6] Chapter II Theorem 4.7). In fact, since X is an affine δ-closed set, let $\mathcal{A}$ be the δ-coordinate ring of X and we can rephrase Theorem 4.4 in the following way:

For any $\mathcal{K} \models T$, $\mathcal{R}$ maximal δ-subring of $\mathcal{K}$ containing $\mathcal{F}$ and for any commutative square as shown, there exists a unique map from $\text{Spec}\mathcal{R}$ to $\text{Spec}\mathcal{A}$ which makes the diagram commute

$$\begin{array}{ccc} \text{Spec}\mathcal{K} & \longrightarrow & \text{Spec}\mathcal{A} \\ \downarrow & \nearrow & \downarrow \\ \text{Spec}\mathcal{R} & \longrightarrow & \text{Spec}\mathcal{F} \end{array}$$

where the maps are induced by δ-homomorphisms between δ-rings.

5 Kolchin's Example: $\mathbb{P}^n(\mathcal{C})$ is Differentially Complete

All the results in this section are due to Kolchin. However, the proofs we give here are slightly different, mainly because we will make use of the model theoretic lemma (3.1) and results in Section 4.

We are going to prove that $\mathbb{P}^n(\mathcal{C})$ is δ-complete. But first we need the following well-known result from commutative algebra ([1] Lemma 5.20).

Lemma 5.1. *Let x be a non-zero element of a field K. Let $(R, \mathfrak{m})$ be a local ring in K. Let $R[x]$ be the subring of K generated by x over R and let $\mathfrak{m}[x]$ be the extension of $\mathfrak{m}$ in $R[x]$. Then either $\mathfrak{m}[x] \neq R[x]$ or $\mathfrak{m}[x^{-1}] \neq R[x^{-1}]$.*

Lemma 5.1 guarantees that we can always extend our δ-homomorphism to a constant point.

Proposition 5.2. *Let $\mathcal{K}, \mathcal{L}$ be δ-fields. Let $x \in \mathcal{K}$ be a constant. Let $f : \mathcal{A} \to \mathcal{L}$ be a δ-homomorphism from a δ-subring $\mathcal{A}$ of $\mathcal{K}$ into $\mathcal{L}$. If $\mathcal{C}_{\mathcal{L}}$, the field of constants of $\mathcal{L}$, is algebraically closed then f extends to a map from either $\mathcal{A}\{x\}$ or $\mathcal{A}\{\frac{1}{x}\}$ to $\mathcal{L}$.*

Proof. The case when $x = 0$ is trivial. So assume $x \neq 0$. Extend f to a maximal δ-homomorphism $\tilde{f} : \mathcal{R} \to \mathcal{L}'$. Since x is a constant, $\mathfrak{m}\{x\} = \mathfrak{m}[x]$ and $\mathfrak{m}\{x^{-1}\} = \mathfrak{m}[x^{-1}]$. By Lemma 5.1, at least one of these is not the unit ideal. Hence, by Proposition 4.3, (2) either x or x^{-1} is in $\mathcal{R}$. Without loss of generality, suppose $x \in \mathcal{R}$. This already shows that f can always be extended either to $\mathcal{A}\{x\}$ or $\mathcal{A}\{x^{-1}\}$ [4]. However, we still have to show that there is an extension of f with range inside $\mathcal{L}$. Let $\mathcal{B} \subset \mathcal{L}$ be the range of f and $\tilde{f}(x) = c$. Consider the maps $\mathcal{A}[x] \to \mathcal{B}[c]$ obtained by restricting $\tilde{f}$ to $\mathcal{A}[x](= \mathcal{A}\{x\})$ and $\mathcal{B}[c] \to \mathcal{L}$ which extends the inclusion by sending c to 0. If c is transcendental over $\mathcal{B}$ then their composition extends f. Otherwise c is algebraic over $\mathcal{B} \subset \mathcal{L}$. Suppose $y^n + l_1 y^{n-1} + \cdots + l_n$ is the minimal polynomial of c over $\mathcal{L}$. Apply δ to $c^n + l_1 c^{n-1} + \cdots + l_n = 0$ and we get $\delta(l_1)c^{n-1} + \cdots + \delta(l_n) = 0$. By minimality of n, $\delta(l_1) = \cdots = \delta(l_n) = 0$. Therefore c is actually algebraic over $\mathcal{C}_{\mathcal{L}}$ which is algebraically closed by the assumption. Thus $c \in \mathcal{C}_{\mathcal{L}}$ and this shows that the range of the extension can always be taken inside $\mathcal{L}$. □

We keep the same notation as in Proposition 5.2.

Corollary 5.3. *Let $f : \mathcal{A} \to \mathcal{L}$ be a δ-homomorphism and let $c_0, \ldots, c_n \in \mathcal{K}$ not all are zero such that $\frac{c_i}{c_j} \in \mathcal{C}$ whenever $c_j \neq 0$. Then there exists $0 \leq j \leq n$ with $c_j \neq 0$ such that f can be extended to a δ-homomorphism from $\mathcal{A}\{\frac{c_0}{c_j}, \ldots, \frac{c_n}{c_j}\}$ to $\mathcal{L}$.*

Proof. We have two cases:
Case 1: Exactly one of the $c_i \neq 0$.
In this case $\mathcal{A}\{\frac{c_0}{c_i}, \ldots, \frac{c_n}{c_i}\} = \mathcal{A}$ so the result is trivial.
Case 2: At least two of the c_i are nonzero.
Without loss of generality $c_n \neq 0$ and $c_i \neq 0$ for some $0 \leq i \leq n-1$. By induction on n and (5.2), we can assume f extends to a δ-homomorphism g from $\mathcal{B} = \mathcal{A}\{\frac{c_0}{c_i}, \ldots, \frac{c_{n-1}}{c_i}\}$ to $\mathcal{L}$. By Proposition 5.2 again, g extends to either $\mathcal{B}\{\frac{c_n}{c_i}\} = \mathcal{A}\{\frac{c_0}{c_i}, \ldots, 1, \ldots, \frac{c_n}{c_i}\}$ or $\mathcal{B}\{\frac{c_i}{c_n}\} = \mathcal{A}\{\frac{c_0}{c_n}, \ldots, \frac{c_{n-1}}{c_n}, \frac{c_i}{c_n}\} \supseteq \mathcal{A}\{\frac{c_0}{c_n}, \ldots, \frac{c_{n-1}}{c_n}, 1\}$. This shows that we can extend f to the required domain in any case. □

Theorem 5.4. *For any $n \in \mathbb{N}$, $\mathbb{P}^n(\mathcal{C})$ is δ-complete.*

Proof. Suppose $Z \subset \mathbb{P}^n(\mathcal{C}) \times \mathbb{A}^m$ is a δ-closed set defined by

$$P_1(y_0, \ldots, y_n; z_1, \ldots, z_m) = \cdots = P_k(y_0, \ldots, y_n; z_1, \ldots, z_m) = 0$$

where $P_i(\bar{y}, \bar{z}) \in \mathcal{F}\{\bar{y}, \bar{z}\}$ is δ-homogeneous in $\bar{y}$ for each $1 \leq i \leq k$. Then $\pi_2 Z$ is defined by $\exists \bar{y}\ \varphi(\bar{y}, \bar{z})$, where $\varphi(\bar{y}, \bar{z})$ is the formula

$$\bigwedge_{i=1}^{k} P_i(\bar{y}, \bar{z}) = 0 \wedge \bigvee_{j=0}^{n} \left(y_j \neq 0 \wedge \bigwedge_{i=1}^{n} \delta(y_i) y_j - y_i \delta(y_j) = 0 \right).$$

[4]Using model completeness, this weaker version is enough to prove Theorem 5.4.

Again suppose we are given $\mathcal{K}, \mathcal{L}, f : \mathcal{A} \to \mathcal{L}$ and $\bar{a}$ as in (3.1). Then there exists $\bar{c} \in \mathcal{K}^{n+1}$ such that not all the c_i's are zero and $\frac{c_i}{c_j}$ is a constant whenever $c_j \neq 0$ and $P_i(c_0, \ldots, c_n, \bar{a}) = 0$ for $1 \leq i \leq k$. By Corollary 5.3, f extends to $\mathcal{A}\{\frac{c_0}{c_j}, \ldots, \frac{c_n}{c_j}\}$ for some $0 \leq j \leq n$. By δ-homogeneity of the P_i's, $P_i(\frac{c_0}{c_j}, \ldots, 1, \ldots \frac{c_n}{c_j}, \bar{a}) = 0$. So $(f(\frac{c_0}{c_j}), \ldots, 1, \ldots, f(\frac{c_n}{c_j}))$ is a solution to $\varphi(\bar{y}, f(\bar{a}))$ in $\mathcal{L}$. □

6 New Examples

So far all the examples we saw are related to the field of constants. From the point of view of model theory, it is natural to ask **whether all the δ-complete sets are nonorthogonal to the constants**. We will give a negative answer to this question in this section.

Definition 6.1. Let $\mathcal{A}$ be a δ-subring of $\mathcal{K}$. An element of $\mathcal{A}\{y\}$ is **monic** if it is of the form $y^n + f(y)$ where the total degree of f (i.e. the degree of f as an element in $\mathcal{A}[y, \delta y, \delta^2 y, \ldots])$ is less than n. An element $x \in \mathcal{K}$ is **monic over $\mathcal{A}$** if x is a zero of some monic δ-polynomial over $\mathcal{A}$.

We will use the following result of Blum [3] in the form that appears in [11].

Proposition 6.2 (Blum).
Let $(\mathcal{R}, \mathfrak{m})$ be a maximal δ-subring of $\mathcal{K}$. Then $x \in \mathcal{K}$ is monic over $\mathcal{R}$ if and only if $x^{-1} \notin \mathfrak{m}$.

The following theorem gives a family of δ-complete sets. Some members of this family are orthogonal to the field of constants.

Theorem 6.3. *Let $P(y)$ be an ordinary polynomial over $\mathcal{F}$. Then the δ-closure in $\mathbb{P}^1$ of the set defined by $\delta y - P(y) = 0$ is δ-complete.*

Proof. Let X be the δ-closed set defined by $\delta y - P(y) = 0$. First consider the case where $\deg P(y) \leq 1$. In this case, $P(y)$ is of the form $ay + b$ where a may be zero. Let y_0 be any point of X. Then $X - y_0$ is the set of solutions of the equation $\delta z = az$. In any case, even if $a = 0$, the solution set is simply the $\mathcal{C}$-span of z_0 where z_0 is a nonzero solution of the equation. Hence the δ-closure of X in $\mathbb{P}^1$ is isomorphic to $\mathbb{P}^1(\mathcal{C})$ which is δ-complete by (5.4). So let us assume $\deg P \geq 2$. By homogenizing the equation, one sees that X is already δ-closed in $\mathbb{P}^1$. Now let $x \in X(\mathcal{K})$ and $\mathcal{R}$ be a maximal δ-subring of $\mathcal{K}$ containing $\mathcal{F}$. By Theorem 4.4, the assertion is true if we can show that $x \in \mathcal{R}$. By (4.3 (2)), it suffices to show that $\mathfrak{m}\{x\} \neq \mathcal{R}\{x\}$. Since $\delta x = P(x)$, $\mathfrak{m}\{x\} = \mathfrak{m}[x]$ and $\mathcal{R}\{x\} = \mathcal{R}[x]$. So by Lemma 5.1, it is enough to show that $\mathfrak{m}[x^{-1}]$ is the unit ideal. Since $\deg P \geq 2$, $a^{-1}(\delta y - P(y))$ is monic where a is the leading coefficient of $P(y)$. It follows from Proposition 6.2 that $x^{-1} \notin \mathfrak{m}$. Therefore, if x^{-1} is in $\mathcal{R}$, so is x. So let us assume that $x^{-1} \notin \mathcal{R}$. By (4.3 (2)) again, we conclude that $\mathfrak{m}\{x^{-1}\}$ is the unit ideal. Moreover, we have $\delta(x^{-1}) = -x^{-2}P(x)$. So there is a nontrivial expression of 1 as:

$$1 = w_{-r}\frac{1}{x^r} + \cdots + w_{-1}\frac{1}{x} + w_0 + w_1 x + \cdots + w_s x^s \tag{1}$$

where $w_i \in \mathfrak{m}$. We move w_0 to the other side of the equation. Note that $1 - w_0$ is a unit in $\mathcal{R}$, so dividing both sides by $1 - w_0$, we get the equation

$$1 = m_{-r}\frac{1}{x^r} + \cdots + m_{-1}\frac{1}{x} + m_1 x + \cdots + m_s x^s \quad (m_i \in \mathfrak{m}) \tag{2}$$

Applying δ to both sides of (2) we get

$$\begin{aligned} 0 &= \delta(m_{-r})\frac{1}{x^r} + r m_{-r}\frac{1}{x^{r-1}}\Big(-\frac{P(x)}{x^2}\Big) + \cdots + \delta(m_{-1})\frac{1}{x} + \\ &\quad + m_{-1}\Big(-\frac{P(x)}{x^2}\Big) + \cdots + \delta(m_s)x^s + s m_s x^{s-1}P(x) \\ 0 &= \text{ lower degree terms } + s a m_s x^{s+d-1} \end{aligned} \tag{3}$$

where $d = \deg P \geq 2$. The important point here is that the coefficient of the highest degree term is a unit times m_s. Divides (3) by sax^{d-1} and eliminates $m_s x^s$ with (2) we get

$$1 = n_{-t}\frac{1}{x^t} + \cdots + n_0 + \cdots + n_k x^k \tag{4}$$

where $k < s$. Since $\mathfrak{m}$ is a δ-ideal, (cf. 4.3 (1)) all the n_i's are still in $\mathfrak{m}$. Again by performing the same trick to (4) as we did to (1), we can assume $n_0 = 0$. Thus we get a similar expression for 1 with degree less than s. So by iterating this process, we conclude that $\mathfrak{m}[x^{-1}]$ is the unit ideal. This finishes the proof. □

Remark 6.4. If the degree of P is equal to 2, then $\delta y = P(y)$ is a Riccati equation. In [5], Cassidy points out that for each Riccati equation there is a projective transformation of $\mathbb{P}^1$ which maps the corresponding δ-variety onto $\mathbb{P}^1(\mathcal{C})$. In particular, we see that the Riccati varieties are δ-complete by (5.4) as well.

Example 6.5. Let X be the δ-closed set defined by $\delta y = y^3 - y^2$. It is δ-complete, strongly minimal and orthogonal to the constants. We will sketch a proof here and refer our readers to [10] for the details. By Theorem 6.3, we know that X is δ-complete. Since X is defined by an order one equation, $\mathrm{RM}(X) = 1$. The δ-ideal $\mathcal{I}$ generated by $\delta y = y^3 - y^2$ is prime since the quotient of $\mathcal{F}\{y\}$ by $\mathcal{I}$ is clearly isomorphic to $\mathcal{F}[y]$. From this we conclude that X is δ-irreducible. So X is strongly minimal.

It remains to argue that X is orthogonal to the constants. Let a be the generic point of X over $\mathcal{F}$. In particular, a is transcendental over $\mathcal{F}$. Let us recall that $\mathcal{C}$ is an algebraically closed field. So by (6.12) in [10], $\mathcal{C}_{\mathcal{F}(a)} = \mathcal{C}$. In fact, we have $\mathcal{C}_{\mathcal{F}\langle a\rangle} = \mathcal{C}$ since $\mathcal{F}\langle a\rangle = \mathcal{F}(a)$ as a satisfies $\delta y = y^3 - y^2$. Let $\mathcal{K}$ be the differential closure of $\mathcal{F}\langle a\rangle$. By Lemma 7.3 in [10], to show that X is orthogonal to the constants all we have to show is that $\mathcal{K}$ has no new constants. By Lemma 2.1 in [10], we have $\mathcal{C}_{\mathcal{K}}$ algebraic over $\mathcal{C}_{\mathcal{F}\langle a\rangle} = \mathcal{C}$ which is algebraically closed, hence $\mathcal{C}_{\mathcal{K}} = \mathcal{C}$. This completes the proof.

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